Hydrodynamics: the other expansion parameter

T.Burch

Universität Regensburg

G. Torrieri

EMMI

HIC for FAIR Helmholtz International Center

FIAS Frankfurt Institute for Advanced Studies

Based on Phys.Rev. D85 (2012) 065006 (GT) ongoing work with Tommy Burch (presented at Lattice2013).

Heavily influenced by work of 

(Related to talks of Kodama/Koide, Hirano at this conference)

“Lower limits” on viscosity

Danielewicz and Gyulassy used the uncertainity principle and Boltzmann equation

$$\eta \sim \frac{1}{5} \langle p \rangle n l_{mfp} \quad , \quad l_{mfp} \sim \langle p \rangle^{-1}$$

but In strongly coupled system the Boltzmann equation is inappropriate

KSS and extensions from AdS/CFT (actually any Gauge/gravity)

$$\eta/s = 1/(4\pi)$$

but theories not realistic, “highly symmetryc”/[athological (the UV-completion is conformally invariant and strongly coupled).

Is there a more general and intuitive way of thinking about these things?

A prelude: Kovtun, Moore, Romatschke, 1104.1586

Basic idea: Kubo formula measures "IR limit" of correlation between energy momentum tensors. As $\eta \to 0$ sound waves can travel to this limit and alter correlation. ("infinite propagation of soundwaves" inpacts "IR limit of Kubo formula").

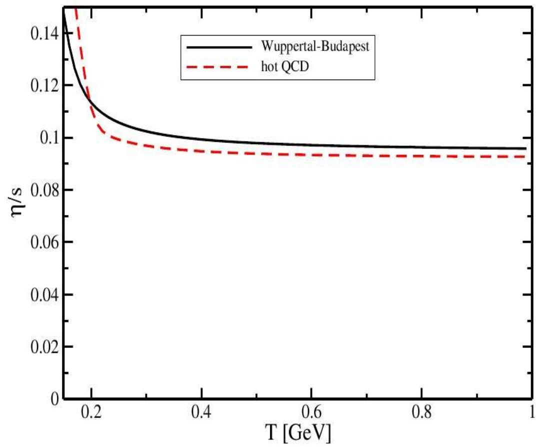
$$G^{xyxy}_{\mathrm{R,shear-shear}}(\omega) \simeq -i\omega \frac{7Tp_{max}}{60\pi^2\gamma_\eta} + (i+1)\omega^{\frac{3}{2}} \frac{7T}{240\pi\gamma_\eta^{\frac{3}{2}}}$$

where p_{max} is the maximum momentum scale and $\gamma_\eta = \eta/(e+p)$

$$\eta_{reno} = \eta_{bare} + \frac{17 p_{max} \frac{\eta_{bare}}{e+p} T(\epsilon+P)^2}{120\pi^2\eta_{bare}^2}$$

Given a p_{max} , $\eta_{bare} \to 0 \Rightarrow \eta_{reno} \to p_{max}/\eta \to \infty$.

Kovtun,Moore and Romatschke plug in p_{max} in terms of $p_{max} \sim (\tau_\pi)^{-1} \sim \eta/(Ts)$. Putting in $\eta_{bare} = s/(4\pi)$ and lattice EoS, they get



G.Moore,P.Romatschke

Phys.Rev.D84:025006,2011 arXiv:1104.1586

Nc=3, η/ s=KSS

This however, "assumes what you are trying to prove": If there is a "microscopic length", you will eventually get a viscosity. Moreover, p_{max} effect on s neglected, hence <u>no</u> renormalized η/s at $p_{max} \to \infty$. What is p_{max} in a generic strongly coupled theory?

is there a more general definition of p_{max}, independent of any dissipative length? Well, let us take a cue from AdS/CFT

The effect described below is not in classical supergravity: sound-waves always $\sim N_c^0$, microscopic degrees of freedom $\sim N_c^2$

$$\frac{\eta}{s} \sim \frac{N_c^2 + \mathcal{O}\left(N_c^0\right)}{4\pi N_c^2}$$

But the theory might be strongly coupled and conformal, $N_c = 3 \ll \infty$. Could p_{max} have something to do with the microscopic rather than dissipative length scale?

Landau and Lifshitz (also D.Rishke,B Betz et al): Hydrodynamics has <u>three</u> length scales

$$\underbrace{l_{micro}}_{\sim s^{-1/3}, n^{-1/3}} \ll \underbrace{l_{mfp}}_{\sim \eta/(sT)} \ll L_{macro}$$

Weakly coupled: Ensemble averaging in Boltzmann equation good up to $\mathcal{O}\left((1/\rho)^{1/3}\partial_\mu f(...)\right)$
Strongly coupled: classical supergravity requires $\lambda \gg 1$ but $\lambda N_c^{-1} = g_{YM} \ll 1$ so

$$\frac{1}{TN_c^{2/3}} \ll \frac{\eta}{sT} \quad \left(\quad or \quad \frac{1}{\sqrt{\lambda}T}\right) \ll L_{macro}$$

QGP: $N_c = 3 \ll \infty$,so $l_{micro} \sim \frac{\eta}{sT}$. Cold atoms: $l_{micro} \sim n^{-1/3} > \frac{\eta}{sT}$?

Why is $l_{micro} \ll l_{mfp}$ necessary? Without it, microscopic fluctuations (which come from the finite number of DoFs and have nothing to do with viscosity) will drive fluid evolution.

$\Delta\rho/\rho \sim C_V^{-1} \sim N_c^{-2}$, thermal fluctuations "too small" to be important!

But we know this approximation is far from perfect

- $N_c = 3 \ll \infty$
- $dN/dy \sim 10^{1-3} \ll \infty$

Lagrangian hydrodynamics (Kodama and others): "Many" particles flowing together in a "small" volume cell. But what does a strongly coupled theory with "finitely many" DoFs look like? How to describe? (non)linear hydrodynamics coupled driven by microscopic fluctuations? QM might help!

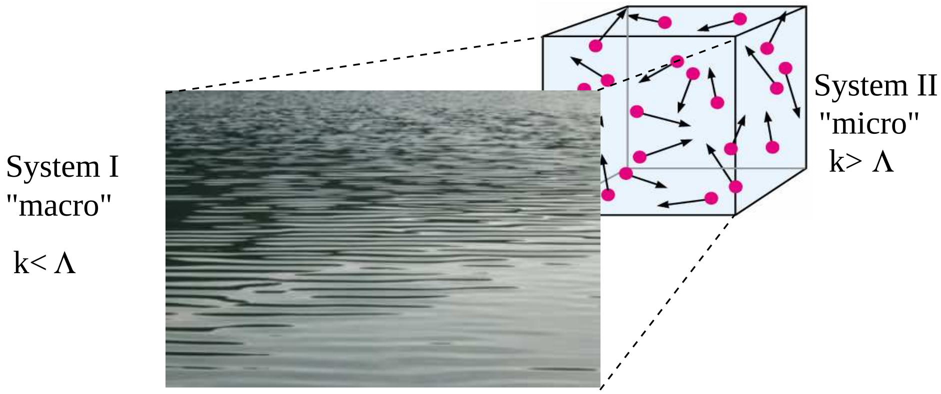
How low can viscosity go if $\Lambda \to \infty$? In the limit where viscosity is so low that soundwaves

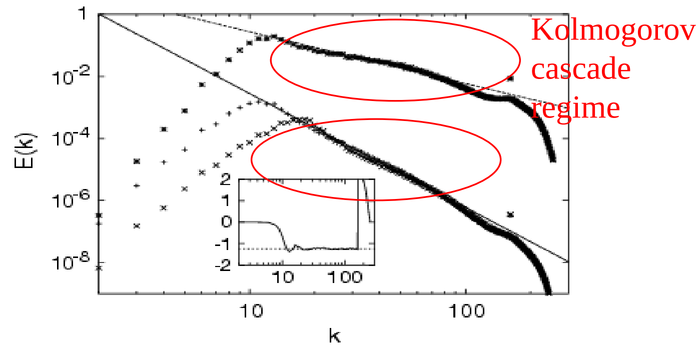
Of amplitude so that momentum $P_{sound} \sim (area)\lambda\,(\delta\rho)\,c_s \gg T$

And wavenumber $k_{sound} \sim P_{sound}$

Survive (ie their amplitude does not decay to $E_{sound} \sim T$) $\tau_{sound} \gg 1/T$

Quantum corrections to sound will be non-negligible, And in "conventional widsom" its not clear how to deal with this!
Is it relevant to physics? good question!

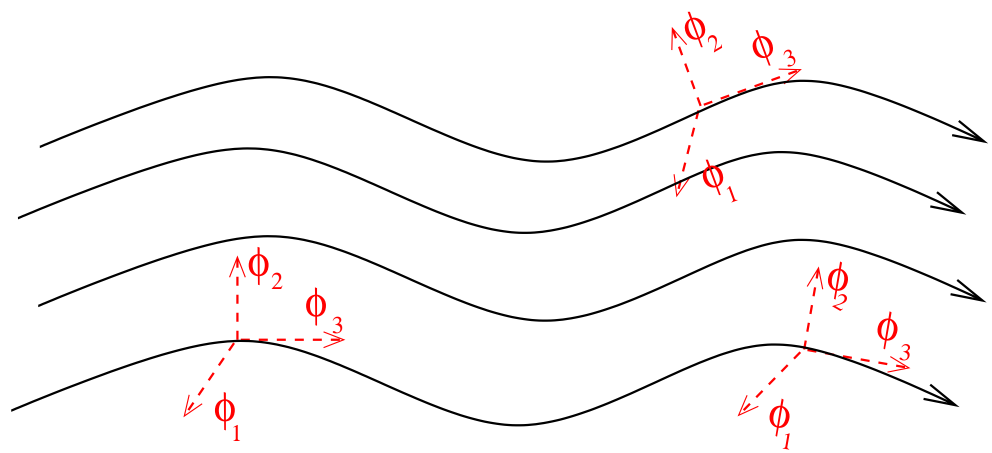




A classical low-viscosity fluid is <u>turbulent</u>. Typically, low-k modes cascade into higher and higher k modes via sound and vortex emission (phase space looks more "fractal"). Classically this process goes on until dissipation, $k \sim \eta/(Ts)$. By essneitally dimensional analysis, Kolmogorov has showsn that provided $\eta/(sT) \ll L_{eddy} \ll L_{boundary}$, $E(k) \sim \left(\frac{dE}{dt}\right)^{2/3} k^{-5/3}$
For a classical ideal fluid, it can go on forever, since $\delta E(k) \sim \delta\rho k c_s$ can be arbitrarily small for arbitrarily high k by making $\delta\rho$ even smaller. but for <u>quantum</u> perturbations, $E \geq k$ so conservation of energy <u>has</u> to cap cascade. A quantum viscosity!

Hydro as fields: (Nicolis et al,1011.6396 (JHEP))

Continuus mechanics (fluids, solids, jellies,...) is written in terms of 3-coordinates $\phi_I(x^\mu), I = 1...3$ of the position of a fluid cell originally at $\phi_I(t = 0, x^i), I = 1...3$. (Lagrangian hydro . NB: no conserved charges)



The system is a Fluid if it's Lagrangian obeys some symmetries (Ideal hydrodynamics $\leftrightarrow$ Isotropy in comoving frame) Excitations (Sound waves, vortices etc) can be thought of as "Goldstone bosons", arising when a theory is expanded around a classical solution.

Translation invariance at Lagrangian level $\leftrightarrow$ Lagrangian can only be a function of $B^{IJ} = \partial_\mu \phi^I \partial^\mu \phi^J$ Now we have a "continuus material"!

Homogeneity/Isotropy means the Lagrangian can only be a function of $B = \det B^{IJ}, \mathrm{diag} B^{IJ}$
The comoving fluid cell must not see a "preferred" direction $\Leftarrow SO(3)$ invariance

Invariance under Volume-preserving diffeomorphisms means the Lagrangian can only be a function of B
In all fluids a cell can be infinitesimally deformed
(with this, we have a fluid. If this last requirement is not met, Nicolis et all call this a "Jelly")

A few exercises for the bored public Check that L = F(B) leads to

$$T_{\mu\nu} = (P+\rho)u_\mu u_\nu - P g_{\mu\nu}$$

provided that

$$\rho = F(B) \; , \qquad p = F(B) - 2F'(B)B \; , \qquad u^\mu = \frac{1}{6\sqrt{B}} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{IJK} \, \partial_\alpha \phi^I \partial_\beta \phi^J \partial_\gamma \phi^K$$

Equation of state chosen by specifying $F(B)$. "Ideal": $\Leftrightarrow F(B) = B^{4/3}$
$\sqrt{B}$ is identified with the entropy and $\sqrt{B}\frac{dF(B)}{dB}$ with the microscopic temperature. You can also show that

$$\partial_\mu \sqrt{B} u^\mu = 0 \quad , \quad s = -\frac{dP}{dT} = \frac{p+\rho}{T}$$

Ie, $\sqrt{B}$ is the conserved quantity corresponding to our earlier group.

Ideal hydrodynamics and the microscopic scale

The most general Lagrangian is

$$L = T_0^4 F\left(\frac{B}{T_0^6}\right) \quad , \quad B = T_0^6 \det B^{IJ} \quad , \quad B^{IJ} = \left|\partial_\mu \phi^I \partial^\mu \phi^J\right|$$

Where $\phi^{I=1,2,3}$ is the comoving coordinate of a volume element of fluid.

NB: $T_0 \sim \Lambda g$ microscopic scale, includes thermal wavelength <u>and</u> $g \sim N_c^2$ (or μ/Λ for dense systems). $T_0 \to \infty \Rightarrow$ classical limit
It is therefore natural to identify T_0 with the microscopic scale!
At $T_0 < \infty$ quantum and thermal fluctuations can produce sound waves and vortices, "weighted" by the usual path integral prescription!

$$\mathcal{Z} = \int \mathcal{D}\phi_i \exp\left[-T_0^4 \int F(B) d^4x\right], \langle \mathcal{O} \rangle \sim \frac{\partial \ln \mathcal{Z}}{\partial ...} \left(eg. \quad \left\langle T_{\mu\nu}^x T_{\mu\nu}^{x'} \right\rangle = \frac{\partial^2 \ln \mathcal{Z}}{\partial g_{\mu\nu}^x \partial g_{\mu\nu}^{x'}} \right)$$

And we discover a fundamental problem: Vortices carry arbitray small energies but stay put! No S-matrix in hydrostatic solution!

$$L_{linear} = \underbrace{\dot{\vec{\pi}}_L^{\,2} - c_s^2(\nabla.\vec{\pi}_L)^2}_{sound\quad wave} + \underbrace{\dot{\pi_T}^2}_{vortex} + Interactions$$

Unlike sound waves , Vortices can not give you a theory of free particles, since they do not propagate: They carry energy and momentum but stay in the same place! Can not expand such a quantum theory in terms of free particles.

Physically: "quantum vortices" can live for an arbitrary long time, and dominate any vacuum solution with their interactions. This does not mean the theory is ill-defined, just that its strongly non-perturbative!

A perturbative attempt GT,Phys.Rev. D85 (2012) 065006

Give "quantum vortices" a propagation speed, $E = c_T p$. Equivalent to modifying Lagrangian to

$$F(B) \to F(B) + \frac{1}{2} c_T^2 B_{II} (= \partial_\mu \phi^I \partial^\mu \phi^I)$$

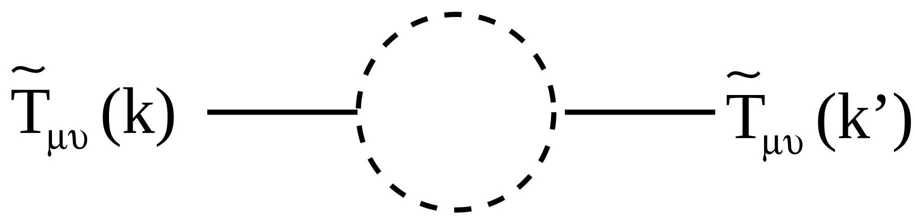
A quantum jelly, $\to$ fluid as $c_T \to 0$ Cross-sections have been computed using these Feynman rules (NB: singular at $c_T \to 0$.

$$\sigma_{sound-sound} = \frac{1}{256\pi} \left(\frac{13}{15} \right) \frac{1}{p^2} \left(\frac{p^4}{w_0 c_T} \right)^2 , \sigma_{sound-vortex} \sim \frac{c_T}{c_s} \frac{1}{p^2} \left(\frac{p^4}{w_0 c_s} \right)^2 \ldots$$

where p is the exchanged momentum, w_0 is the *microscopic* enthalpy density $(= Ts)$ of the background fluid and $\alpha \equiv (f_4/c_s^2 - 2f_3^2/c_s^4 + 3c_s^2 + 2f_3 + c_s^4) = \mathcal{O}(1) + \mathcal{O}(c_s^2) + \mathcal{O}(c_s^4)$.

$$\langle T_{xy}(x)T_{xy}(x')\rangle = \mathcal{N}\int \mathcal{D}B\,(T_{xy}(x)T_{xy}(x'))\exp\left[-iT_0^4\int F(B)d^4x\right]$$

find IR limit and compare it with Kubo formula!. Can be done by Feynman diagram techniques. “small parameter” the speed of sound.



S.Jeon (PRD **52**, 3591 (1995)) showed us that to tree level the Kubo formula viscosity is exactly equivalent to “Lifshitz-Landau formula” $\eta = \#\langle p\rangle\langle n\rangle l_{mfp}$, $l_{mfp} = \frac{1}{\langle\langle n\sigma\rangle\rangle}$ NB This breaks unitarity. unsurprising: Quantum theories can have anomalies, where the UV scale (T_0^4) breaks “fundamental” IR symmetries. Unitarity broken order-by-order in σ-models

The leading order result

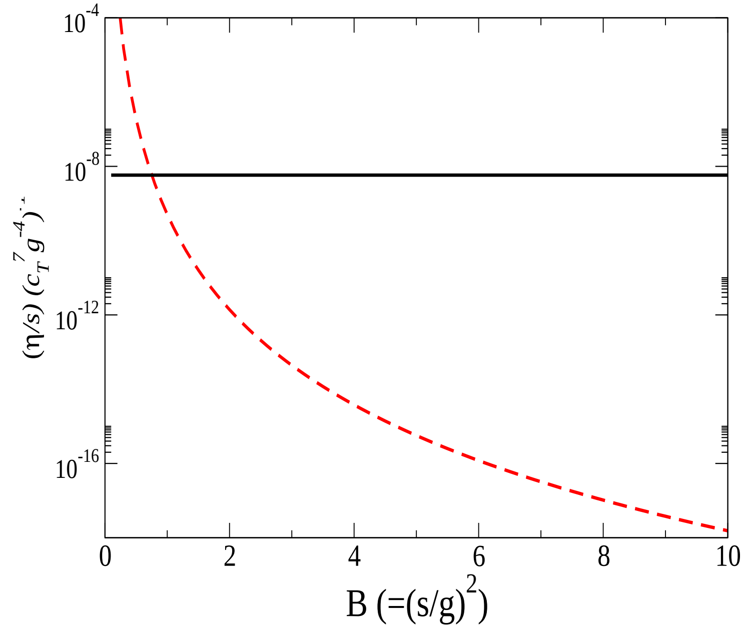
$$\frac{\eta}{s} = K_0 \frac{c_T^{14} g^8}{B^2 (dF/dB)^6}, K_0 = \frac{\zeta(3)^2 \zeta(9)}{80640} \frac{4}{256\pi} \frac{13}{45} \frac{\pi^2}{15} \left(\frac{4\pi^4}{45} \right)^{-1} \simeq 1.96(10^{-9})$$

The meat: If $c_T^{14} g^8 \sim \mathcal{O}(1)$, η/s finite and evolves with EoS.

$$\text{Result for } F(B) = \begin{cases} B^{2/3} & \text{Ideal gas} \\ f_1 B^{\zeta} / B_0^{f_2/2-2/3} & \text{lQCD fit} \end{cases}$$

G.Torrieri

Phys.Rev. D85 (2012) 065006



But we deformed the theory Can we do better? Yes, Lattice!

The non-deformed theory is essentially non-perturbative,but it might be amenable to a lattice treatment

$$\int \mathcal{D}\phi_I \exp\left(i \int d^4xL\right) \underbrace{\rightarrow}_{lattice+Wick} \int d\phi_I^i \exp\left[-(T_0\Delta x)^4 \sum_i F(\phi_i)\right]$$

Continuum limit: $\delta \rightarrow \Delta x T_0 \ll 1$. Study the behaviour of $\lim_{\delta\rightarrow 0} \langle X \rangle$.

Some specific considerations:

recall that $B_{IJ} = \partial_\mu \phi^I \partial^\mu \phi^J$ and $u^\mu = \frac{1}{b}\epsilon^{\mu\alpha\beta\gamma}\partial_\alpha\phi^1\partial_\beta\phi^2\partial_\gamma\phi^3$ and: $\mathcal{L} = F(b)$; $b = \sqrt{\det B_{IJ}}$ to avoid problems with periodic boundaries use "shifted" fields ("subtract" the hydrostatic background)...

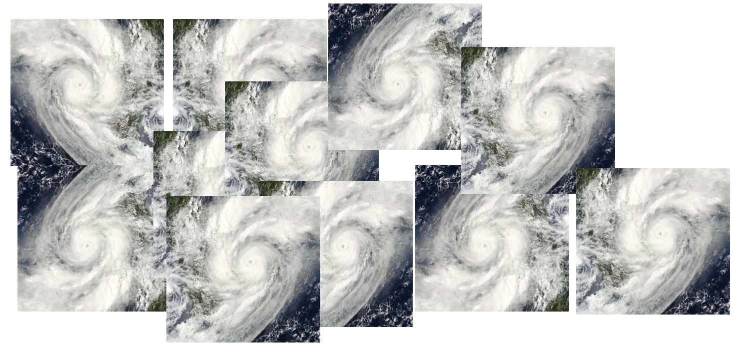
$$\pi^I = \phi^I - x^I \quad \rightarrow \quad \partial_\alpha \phi^I = \partial_\alpha \pi^I + 1\delta_\alpha^I$$

Some interesting observables

- "Scalar perturbation" $\frac{\langle B(dF/dB)\rangle}{\langle T_{\mu}^{\mu}\rangle}$
- "Vector perturbation" $\langle u_\mu u_\nu + g_{\mu\nu}\rangle = \left\langle \frac{1}{B_{IJ}} \partial_\mu \phi^I \partial_\nu \phi^J \right\rangle$
- Vorticity $C_P = \oint_P (p+\rho) u_\mu dx^\mu$

Averages and Fluctuation, correlator,spectral function interesting. Modifications of either with T_0 could indicate transition to "quantum turbulence".

So, what can we calculate?



Non−trivial ground state

Is there a phase transition at a critical T_0 between a "classical" hydrostatic vacuum and a vacuum dominated by quantum/thermal turbulence (bound states of quantum vortices and the like)?

Is the theory trivial in the RG group sense?
What $F(B)$ admit to a well-behaved continuum limit?

These questions can be answered by a lattice calculation. No spurious c_T parameter or perturbative expansion needed

Consider a conformal fluid with no degeneracy and one microscopic DoF

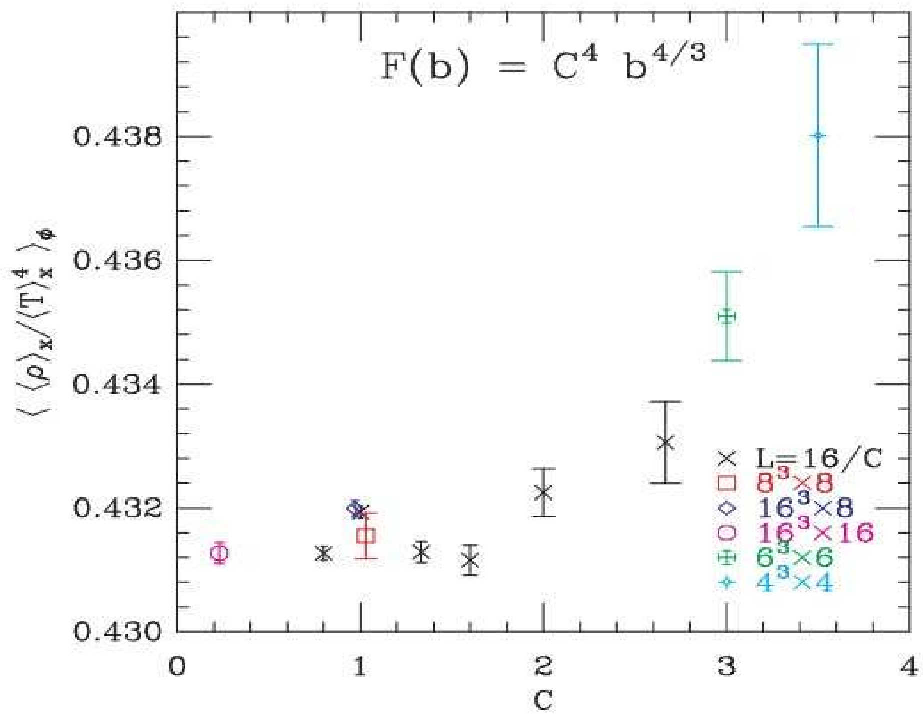
In the classical hydrostatic limit (Where $B=1$)

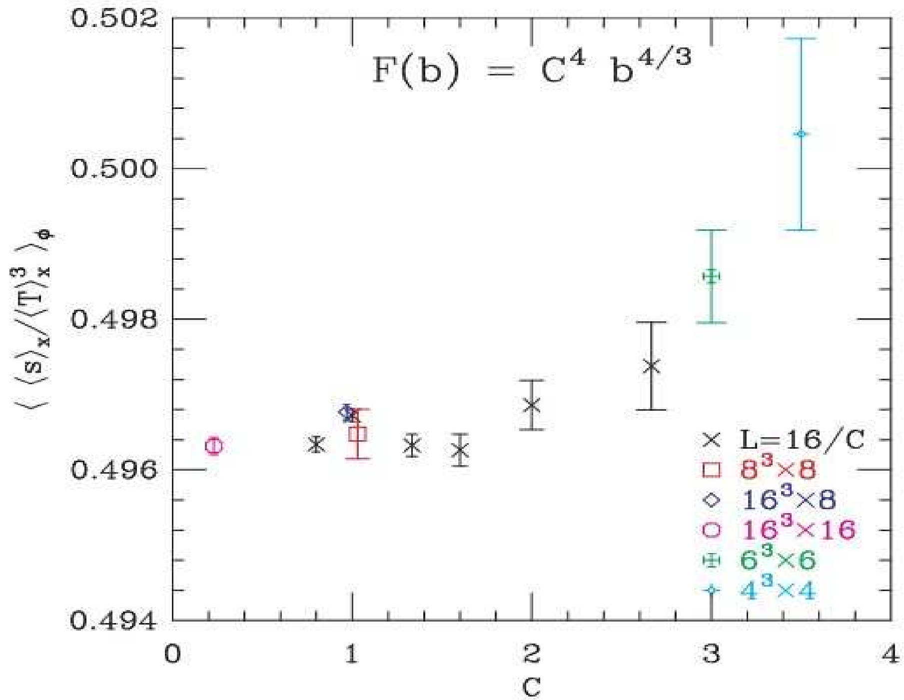
$$\left.\begin{array}{rcl} e = & T_0^4 B^{2/3} & = \frac{g\pi^2}{60} T^4 \\ s = & T_0^3 \sqrt{B} & = \frac{g\pi^2}{45} T^3 \\ T = & \frac{e+p}{s} & = \frac{4}{3g} T_0 B^{1/6} \end{array}\right\} T = \frac{4}{3g} T_0 = \frac{\chi}{a} ?$$

where g is the microscopic degeneracy$\sim N_c^2$. And of course

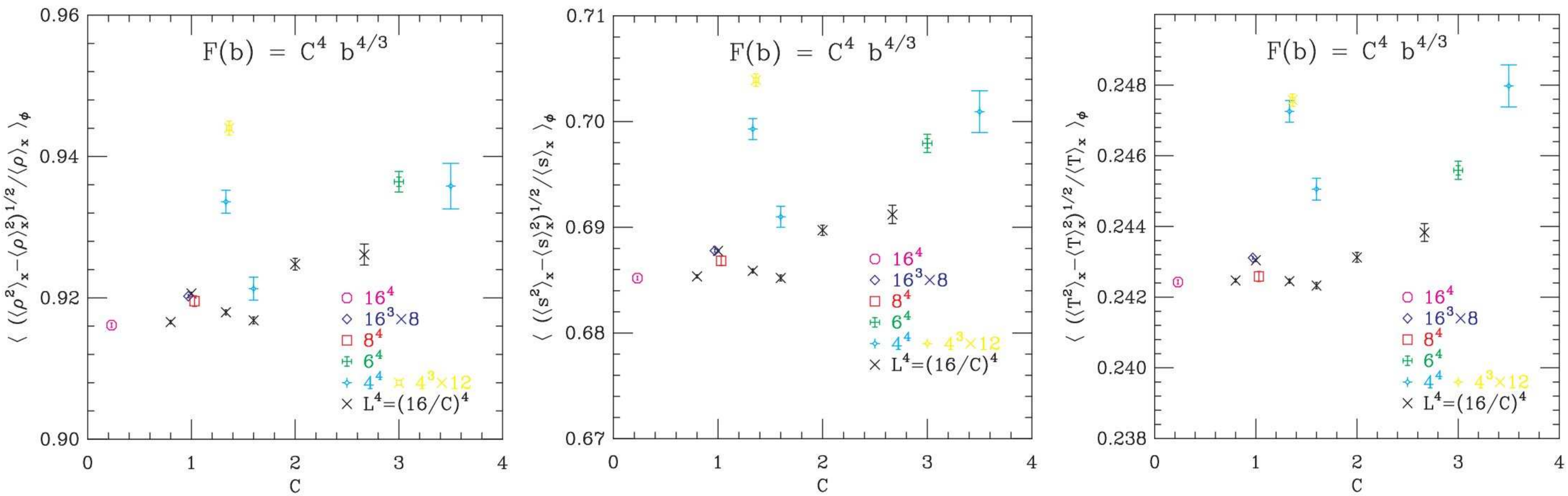
$$\vec{u^\mu} = (1,\vec{0}), \langle u_\mu u_\nu \rangle = \delta_{00} \quad , \quad \langle T_{\mu\nu} \rangle = \frac{\delta \ln Z}{\delta g_{\mu\nu}} \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & e/3 & 0 & 0 \\ 0 & 0 & e/3 & 0 \\ 0 & 0 & 0 & e/3 \end{pmatrix}$$

with higher order correlations vanishing. If quantum vacuum non-trivial $T_{\mu\nu} = (p+\rho) u_\mu u_\nu + p g_{\mu\nu}$ so $e = F(B) \neq T_{00}, p = B\frac{dF(b)}{dB} \neq T_{ii}$ etc.



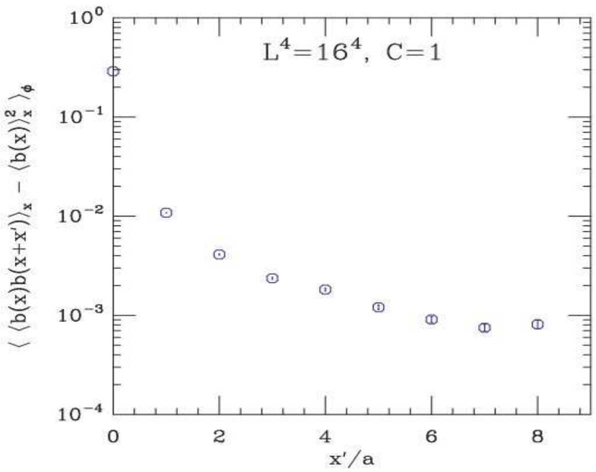


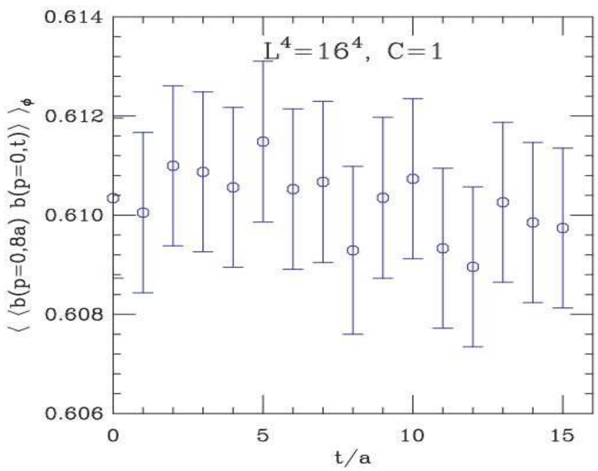
Quantum mechanics means scale a potentially physical "cutoff", dominating dynamics $aT_0 \sim C$. interesting structure at high C... Crossover to collective-dominated regime or lattice artifact?

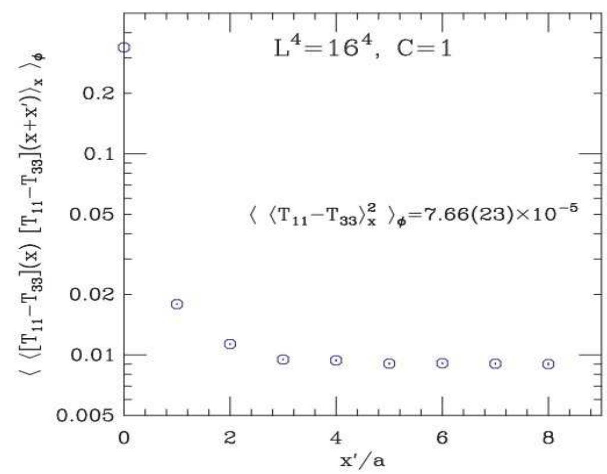


Normalized fluctuations independent of C, but the constants of proportionality non-trivial

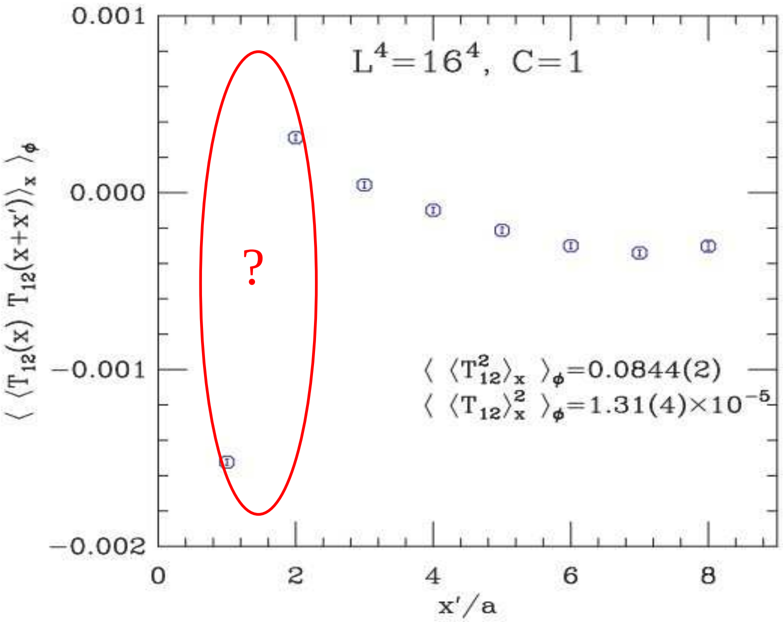
Fluctuations high Are we "missing" phase transition by measuring average observables? Are fluctuations part of "new phase" ?

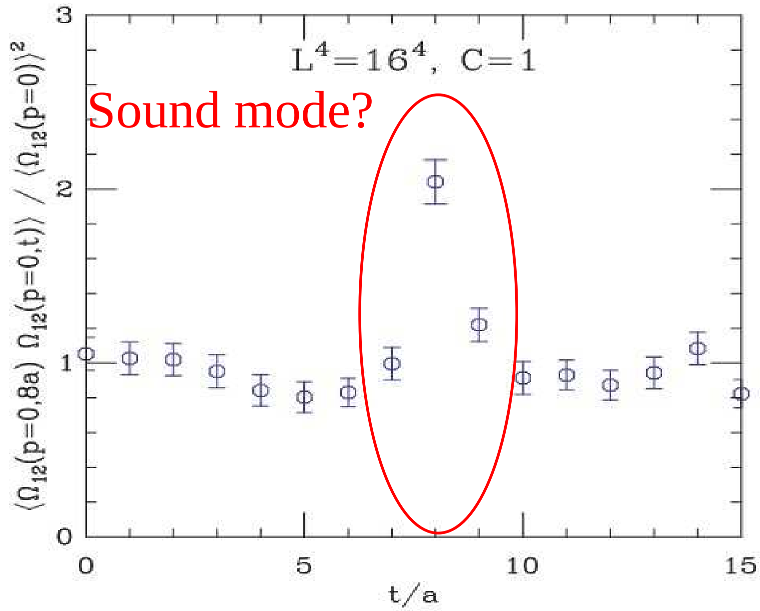






Entropy and energy density correlators, “quantum corrections” to equations of state?





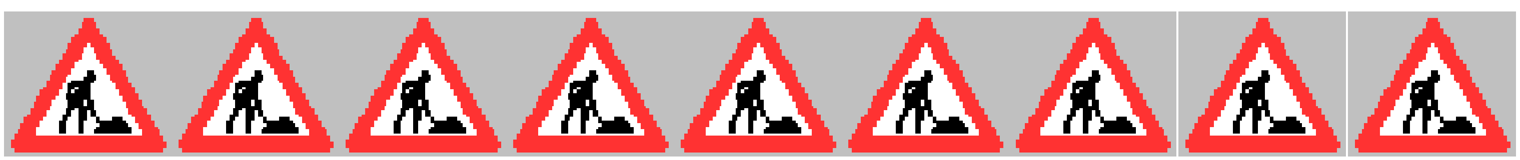
Off diagonal and diagonal elements have long-time correlation. To what extent is this "similar to a quantum viscosity" ?

Instead of a conclusion: further steps

Understand the continuum limit How do observables converge when it is approached?

- Langevan semi-classical limit? (Relativistic generalization of T.Koide, T.Kodama, 1105.6256)
- Under what circumstances is the hydrostatic limit stable?
- If vacuum non-trivial, what are the effective degrees of freedom? (e.g. Enstrophy/Vortex crystal in 2D? Breakdown in translational invariance easy to see in simulation!)

Connecting to "usual" transport BBGKY hyerarchy, gradient expansion



Spare slides

$T_0 \to \infty$ limit on the lattice

If $\lim_{\delta\to 0} \langle X \rangle \sim \langle X_0 \rangle_{termostatic}$, Thermostatic state is stable.

If $\lim_{\delta\to 0} \langle X \rangle / \langle X_0 \rangle \sim f(B)$, vacuum non-trivial but "well-behaved".

If $\lim_{\delta\to 0} \langle X \rangle / \langle X_0 \rangle \sim \delta^{-\alpha}$ or $\sim \exp(\alpha\delta^{-1})$ for universal degrees of diverge α, the theory is renormalizeable: δ is needed to set an absolute scale, but dimensionless ratios are independent of it.

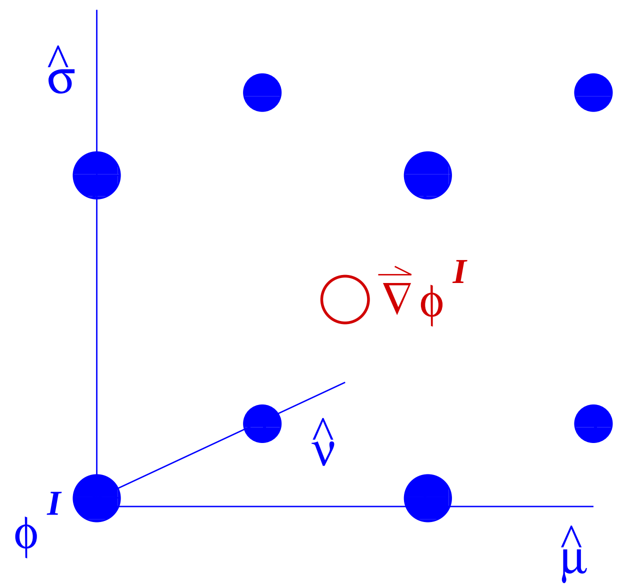
If $\lim_{\delta\to 0} \langle X \rangle / \langle X_0 \rangle \sim \delta^{-\alpha}$ or $\sim \exp(\alpha\delta^{-1})$ for αs that are $\langle X \rangle$-specific (One α for the scalar and another for the tensor,defined below) the theory is "trivial" , in that taking $\delta \to 0$ makes the vacuum diverge. In this case, step **(ii)** of the previous section is strictly impossible.

since we expect extended structures (e.g., vortices) we use HMC updates: one therefore needs the variation of the action w.r.t. the local field values...

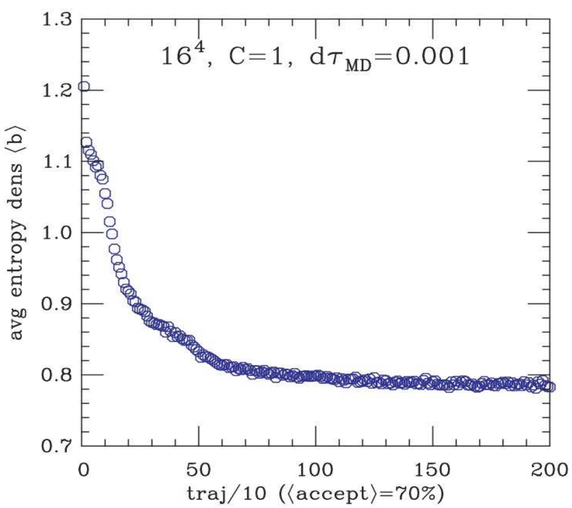
$\frac{\delta S}{\delta \phi^I(x)} = \frac{\delta S}{\delta b} \frac{\delta b}{\delta(\partial_\alpha \phi^J)} \frac{\delta(\partial_\alpha \phi^J)}{\delta \phi^I(x)} =$

$\sum_{y,\mu,\nu,\sigma} \frac{dF}{db} \delta^{IJ} \delta(y - x \pm \hat{\mu}/2 \pm \hat{\nu}/2 \pm \hat{\sigma}/2) \quad \frac{b}{8} B^{-1}_{JK} \left|\epsilon_{\mu\nu\sigma\alpha}\right| \partial_\alpha \phi^K \Big|^{y-\hat{\alpha}/2}_{y+\hat{\alpha}/2}$

fields (ϕ^I) occupy lattice sites; derivatives (and hence B_{IJ}, b, u_μ, $T_{\mu\nu}$, etc.) defined at body centers of hypercubes:



L^4	$C(=aT_0)$	traj	$d\tau_{MD}$	accept
20^4	0.8	4000	0.001 / 0.0005	49% / 85%
16^4	1	10000	0.001	52%
12^4	1.33333	10000	0.0005	61%
10^4	1.6	10000	0.0005	41%
8^4	2	10000	0.00025	72%
6^4	2.66667	10000	0.00025	56%



C/Open MP code

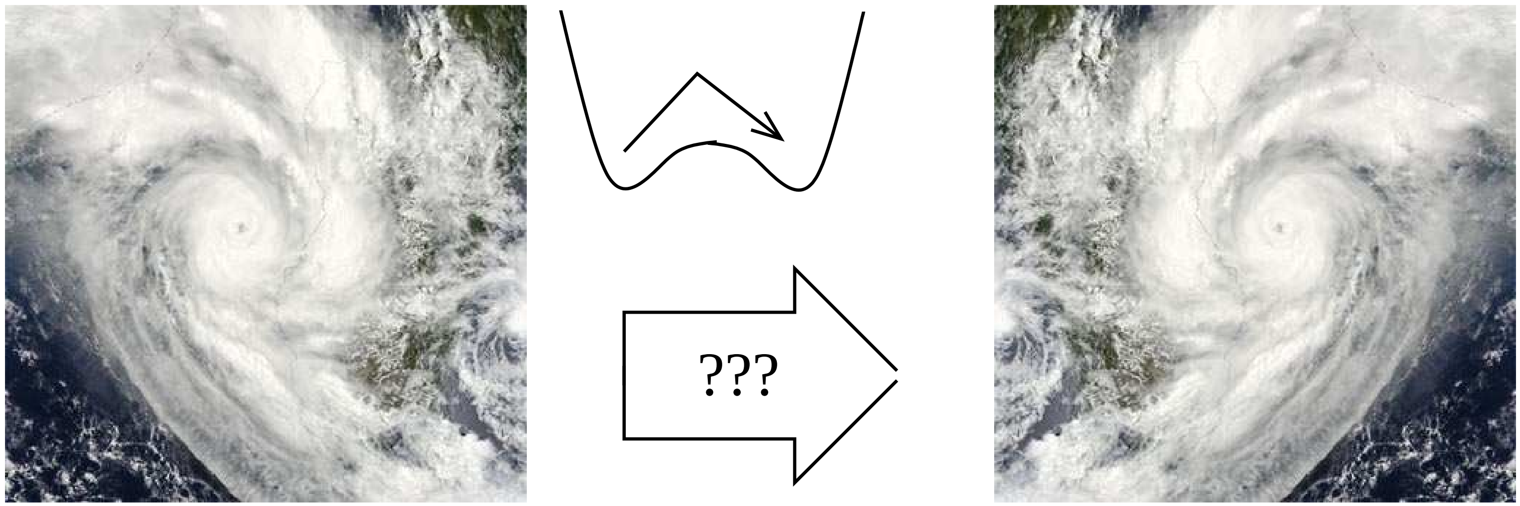
HMC algorithm

Runs with constant physical volume

$L^4=(16/C)^4$

C: assumed fundamental scale $\sim aT_0$

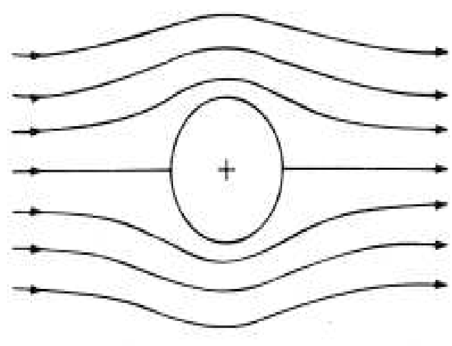
Non-perturbative dissipation loss ("quantum" turbulence)



Classical hydrodynamics has infinitely many solutions arbitrarily close together.
Could WKB-type jumps among solutions with different entropy content be allowed? work in progress!

Example: The Dalambert problem

Analytical solution Euler equation

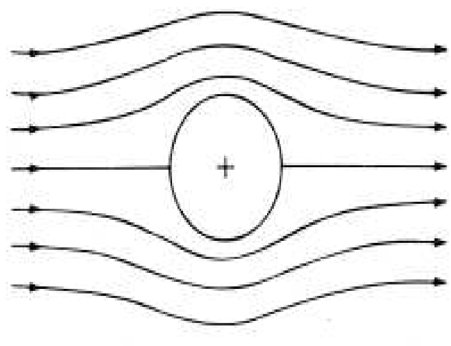


Cylinder and asymptotic flow

Analytically solvable:

$$v_r = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta \quad , \quad v_\theta = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta$$

Analytical
solution
Euler equation

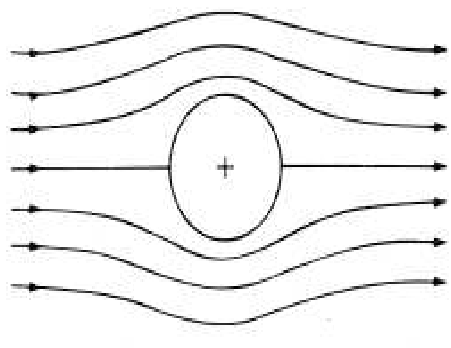


Cylinder
and
asymptotic flow

For Energy to be the same $\frac{\rho(U_1)}{\rho(U_2)} = \left(\frac{U_2}{U_1}\right)^2$

NB: Entropy density <u>different</u> for each U

Analytical
solution
Euler equation



Cylinder
and
asymptotic flow

Rewrite in ϕ_I and find minima in $\left\langle \phi^I_{\vec{x}_0,U,\mathcal{E}} \middle| \middle| \phi^I_{\vec{x}_0',U',\mathcal{E}'} \right\rangle \sim \exp\left[-\Delta S_{U,U'}\right]$

$$\Delta S_{U,U'} = \int d^4x \sum_{IJ} \left.\frac{\delta^2 S}{\delta\phi^I\delta\phi^J}\right|_{\phi^{I,J}=\phi^I_{\vec{x}_0,U,\mathcal{E}}} \sum_{IJ} \left(\phi^I_{\vec{x}_0,U,\mathcal{E}} - \phi^I_{\vec{x}'_0,U',\mathcal{E}}\right)\left(\phi^J_{\vec{x}_0,U,\mathcal{E}} - \phi^J_{\vec{x}'_0,U}\right.$$

What does this mean?

Why does a quintessentially unitary theory (quantum mechanics!) set a lower limit to dissipative processes?
How does one reconcile quantum viscosity with Von Neumann's theorem?

$$\frac{d}{dt}\mathrm{Tr}\hat{\rho}\ln\hat{\rho} = 0$$

My tentative answer: Quantum field theory also sets limit to scale beyond which we measure! Quantum correlations in a many particle system inevitably go over that scale.

What the hell does this all mean? II

Loss of unitarity at the renormalization scale. A quantum field with many particles obeys the fully quantum equation of motion

$$\frac{d\hat{\rho}}{dt} = i\left[\mathcal{H}, \hat{\rho}\right]$$

where $\hat{\rho}$ is the <u>density operator</u> for the <u>field</u>

$$\hat{\rho}(x) = \sum_{k,k'} A_{k,k'} a^{+}_{k,k'} |0 >< 0| a_{k,k'}$$

and $\mathcal{H}$ is the Hamiltonian density.

Like all QFT equations, this has to be regulated by a momentum scale Λ (plus, fluid theory non-renormalizable). Generally, information should flow across the cut-off (ie, get lost among the "fast" degrees of freedom), so effective theory <u>dissipative</u>

Conclusions

Ie, what needs to be done before I have a result

Understand divergences Under what circumstances, if any, can g, c_T, p_{max} diverge while η/s is constant

Understanding how does this constrain the "running" of η/s with $\sqrt{B}, c_T$

Understanding whether this makes any sense...

Work in progress... if you think you can help, Id like to hear from you!