Nonlinear waves in strongly interacting relativistic fluids

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Introduction

Hydrodynamics

Equations of state

Linearization

Beyond linearization: RPM

Nonlinear wave equations

Cold hadron matter

Cold quark matter

Hot quark matter

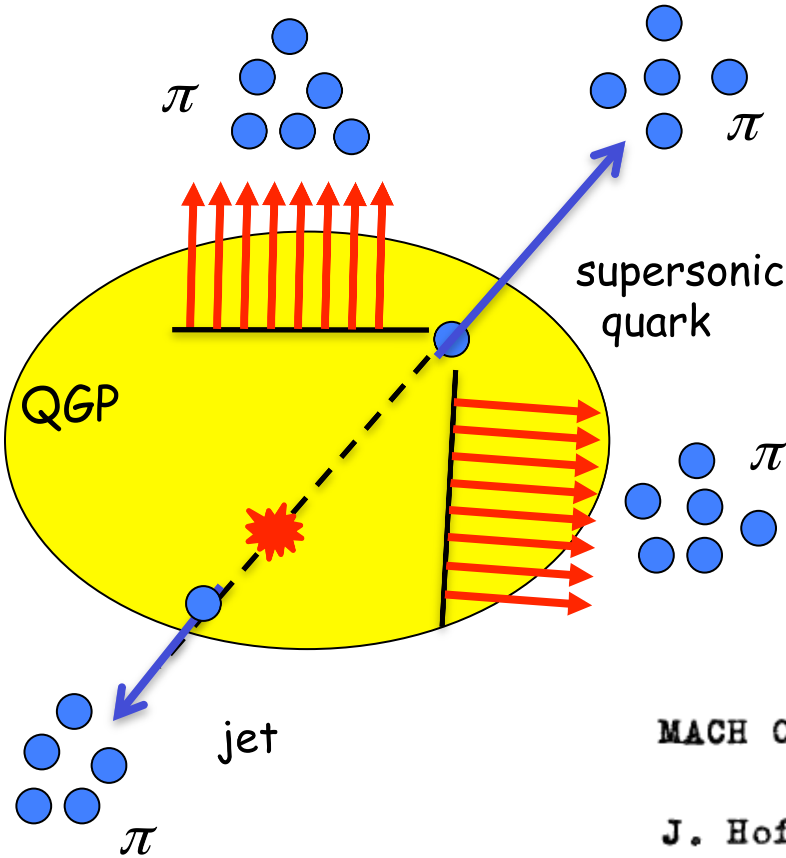
Conclusion

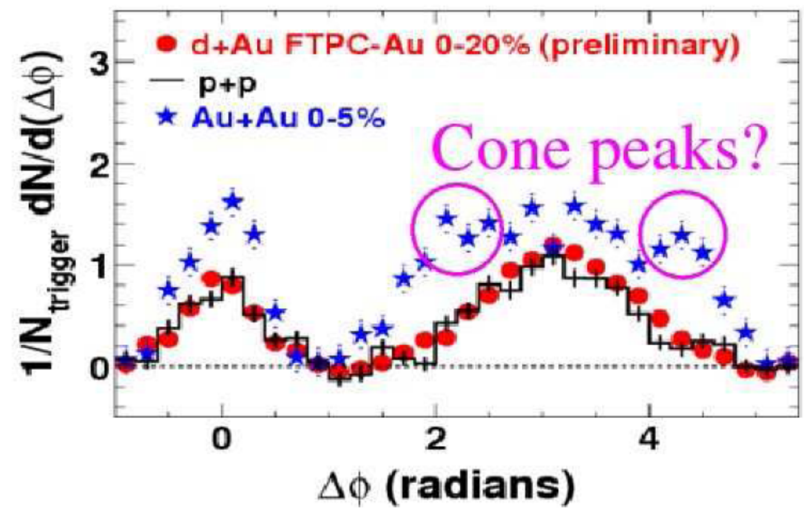
Fogaca, FSN, Ferreira Filho
arXiv:1212.6932

The best
fireball !!!

Happy Birthday Takeshi !

Perturbations in the QGP





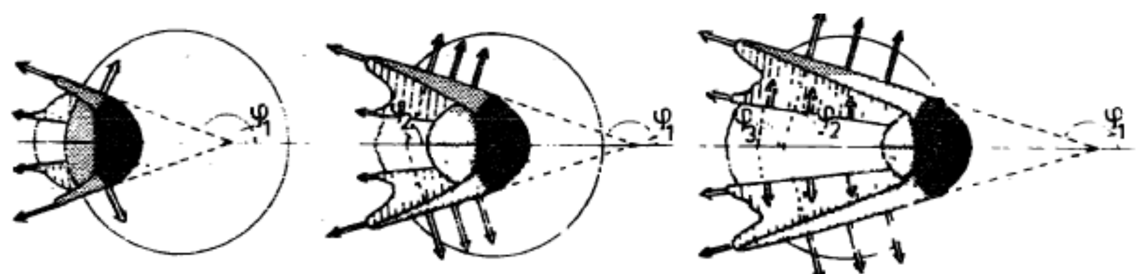
MACH CONES IN FAST NUCLEUS-NUCLEUS COLLISIONS II

1975 !!!

J. Hofmann, H. Stöcker, W. Scheid and W. Greiner
Institut für Theoretische Physik

Waves in the
quark gluon plasma?

Mach cones ?



Relativistic Hydrodynamics

Energy momentum conservation

$$\partial_\nu T_\mu{}^\nu = 0 \qquad T^{\alpha\beta} = (\varepsilon + p)u^\alpha u^\beta - p g^{\alpha\beta}$$

Euler

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{(\varepsilon+p)\gamma^2}\left(\vec{\nabla}p + \vec{v}\frac{\partial p}{\partial t}\right)$$

Baryon number conservation

$$\partial_\nu j_B{}^\nu = 0 \qquad j_B{}^\nu = \rho_B u^\nu$$

Continuity

$$\frac{\partial \rho_B}{\partial t} + \gamma^2 v \rho_B \left(\frac{\partial v}{\partial t} + \vec{v}\cdot\vec{\nabla}v\right) + \vec{\nabla}\cdot(\rho_B \vec{v}) = 0$$

Entropy conservation

$$\partial_\nu (s u^\nu) = 0$$

$$\frac{\partial s}{\partial t} + \gamma^2 v s \left(\frac{\partial v}{\partial t} + \vec{v}\cdot\vec{\nabla}v\right) + \vec{\nabla}\cdot(s\vec{v}) = 0$$

Equation of State

Hadronic matter

Relativistic mean field model (nonlinear Walecka) :

$$\mathcal{L}^* = \bar{\psi}[(i\gamma_\mu\partial^\mu - g_V\gamma_0 V_0) - (M - g_S\phi_0)]\psi + \frac{1}{2}\Big(\partial_\mu\phi_0\partial^\mu\phi_0 - {m_S}^2{\phi_0}^2\Big) +$$

$$+\frac{1}{2}(\vec{\nabla}V_0)^2 + \frac{1}{2}{m_V}^2{V_0}^2 - \frac{b}{3}{\phi_0}^3 - \frac{c}{4}{\phi_0}^4$$

Mean field Lagrangian

$$\begin{cases} -\vec{\nabla}^2 V_0 + {m_V}^2 V_0 = g_V\bar{\psi}\gamma^0\psi \\ (\partial_\mu\partial^\mu + {m_S}^2)\phi_0 = g_S\bar{\psi}\psi - b{\phi_0}^2 - c{\phi_0}^3 \\ \Big[i\gamma_\mu\partial^\mu - g_V\gamma_0 V_0 - (M - g_S\phi_0)\Big]\psi = 0 \end{cases}$$

Equations of motion

$$T^{\mu\nu} = \frac{\partial \mathcal{L}^*}{\partial(\partial_\mu \eta_i)}(\partial^\nu \eta_i) - g^{\mu\nu}\mathcal{L}^* - \left[\partial_\beta \frac{\partial \mathcal{L}^*}{\partial(\partial_\mu \partial_\beta \eta_i)}\right](\partial^\nu \eta_i) + \frac{\partial \mathcal{L}^*}{\partial(\partial_\mu \partial_\beta \eta_i)}(\partial_\beta \partial^\nu \eta_i)$$

$$\longrightarrow \quad \varepsilon =< T_{00} > \qquad p = \frac{1}{3} < T_{ii} >$$

Usually: $\nabla^2 V_0 = 0 \quad \longrightarrow \quad V_0 = \frac{g_V}{{m_V}^2}\rho_B$

Here: $V_0 = \frac{g_V}{{m_V}^2}\rho_B \quad \longrightarrow \quad \nabla^2 V_0 = \frac{g_V}{m_V^2}\nabla^2 \rho_B$

$$\varepsilon = \frac{{g_V}^2}{2{m_V}^2}{\rho_B}^2 + \frac{{g_V}^2}{2{m_V}^4}\rho_B \vec{\nabla}^2 \rho_B + \ldots$$

Quark matter

MIT Bag Model

$$T = 0 \quad \begin{cases} \varepsilon(\rho_B) = \left(\frac{3}{2}\right)^{7/3} \pi^{2/3} {\rho_B}^{4/3} + \mathcal{B} \\ p(\rho_B) = \frac{1}{3}\left(\frac{3}{2}\right)^{7/3} \pi^{2/3} {\rho_B}^{4/3} - \mathcal{B} \end{cases}$$

$$T \neq 0 \quad \begin{cases} \varepsilon = \mathcal{B} + \frac{\gamma_G}{(2\pi)^3} \int d^3k \;\; k \;\; (e^{k/T} - 1)^{-1} + \frac{\gamma_Q}{(2\pi)^3} \int d^3k \;\; k \;\; [n_{\vec{k}} + \bar{n}_{\vec{k}}] \\ p = -\mathcal{B} + \frac{1}{3}\left\{ \frac{\gamma_G}{(2\pi)^3} \int d^3k \;\; k \;\; (e^{k/T} - 1)^{-1} + \frac{\gamma_Q}{(2\pi)^3} \int d^3k \; k \left[n_{\vec{k}} + \bar{n}_{\vec{k}} \right] \right\} \\ s = 4 \frac{37}{90} \pi^2 T^3 \qquad {c_S}^2 = \frac{\partial p}{\partial \varepsilon} = \frac{1}{3} \end{cases}$$

Mean field QCD

Fogaca, FSN,
PLB (2011) arXiv:1012.5266

(attempt to describe a sQGP at T=0)

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_{q=1}^{N_f} \bar{\psi}^q_i \Big[i\gamma^\mu (\delta_{ij}\partial_\mu - igT^a_{ij}G^a_\mu) - \delta_{ij} m_q \Big] \psi^q_j$$

$$F^{a\mu\nu} = \partial^\mu G^{a\nu} - \partial^\nu G^{a\mu} + g f^{abc} G^{b\mu} G^{c\nu}$$

$$\underbrace{G^{a\mu}}_{} = \underbrace{A^{a\mu}}_{\text{soft}} + \underbrace{\alpha^{a\mu}}_{\text{hard}}$$

$\alpha^a_\mu(\vec{x},t) = \delta_{\mu 0}\, \alpha^a_0(\vec{x},t)$ mean field

$A^a_\mu \longrightarrow \; < A^a_\mu \, A^{a\mu} > \quad < A^a_\mu \, A^b_\nu \, A^{a\mu} \, A^{b\nu} >$

condensates

$$m_G^2 = -\frac{9}{32} < A^2 > \qquad < \frac{\alpha_s}{\pi} F^a_{\mu\nu} \, F^{a\mu\nu} >$$

dynamical gluon mass

bag like-term $\mathcal{B}_{QCD}$

$$\mathcal{L}_0 = -\frac{1}{2}\alpha_0^a(\vec{\nabla}^2\alpha_0^a) + \frac{{m_G}^2}{2}\alpha_0^a\alpha_0^a - \mathcal{B}_{QCD} + \bar{\psi}_i\Big(i\delta_{ij}\gamma^\mu\partial_\mu + g\gamma^0 T^a_{ij}\alpha_0^a\Big)\psi_j$$

Equations of motion

$$\begin{cases} -\vec{\nabla}^2\alpha_0^a + {m_G}^2\alpha_0^a = -g\rho^a \\ \Big(i\gamma^\mu\partial_\mu + g\gamma^0 T^a\alpha_0^a\Big)\psi = 0 \end{cases}$$

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\eta_i)}(\partial^\nu\eta_i) - g^{\mu\nu}\mathcal{L} - \left[\partial_\beta\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\beta\eta_i)}\right](\partial^\nu\eta_i) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\beta\eta_i)}(\partial_\beta\partial^\nu\eta_i)$$

EOS

$$\begin{cases} \varepsilon = \left(\frac{27g^2}{16{m_G}^2}\right){\rho_B}^2 + \left(\frac{27g^2}{16{m_G}^4}\right)\rho_B\vec{\nabla}^2\rho_B + \mathcal{B}_{QCD} \\ p = \left(\frac{27g^2}{16{m_G}^2}\right){\rho_B}^2 + \left(\frac{9g^2}{4{m_G}^4}\right)\rho_B\,\vec{\nabla}^2\rho_B - \mathcal{B}_{QCD} \end{cases}$$

Hydrodynamics of perturbations

I) Linearization

Expansion around the equilibrium: $v = 0$

$$\left\{ \begin{aligned} \varepsilon(x,y,z,t) &= \varepsilon_0 + \delta\varepsilon(x,y,z,t) \\ p(x,y,z,t) &= p_0 + \delta p(x,y,z,t) \end{aligned} \right.$$

Neglect:

$$v^2, \;\; v\delta\varepsilon, \;\; v\delta P, \;\; \vec{v}\cdot\vec{\nabla}v, \;\; (\vec{v}\cdot\vec{\nabla})\vec{v}$$

Obtain :

$$\boxed{\frac{\partial^2(\delta\varepsilon)}{\partial t^2} - {c_S}^2\vec{\nabla}^2(\delta\varepsilon) = 0} \qquad c_S = \left(\frac{\partial p}{\partial \varepsilon}\right)^{1/2}$$

linear waves — sound velocity

What if the perturbations are not so small ?

II) Beyond linearization: Reductive Perturbation Method (RPM)

$$\frac{\partial F}{\partial t} + \alpha F \frac{\partial F}{\partial x} = 0$$

Washimi, Taniuti, 1966

Leblond, 2008

Fogaca, FSN, Ferreira Filho
arXiv:1212.6932

$$F(x,t) = F_0 + \bar{F}_1 + \bar{F}_2 + \ldots \qquad F_0 = const$$

$$F(x,t) = F_0 + \sigma F_1(x,t) + \sigma^2 F_2(x,t) + \sigma^3 F_3(x,t) + \ldots \qquad \sigma < 1$$

$$\sigma \frac{\partial F_1}{\partial t} + \sigma^2 \frac{\partial F_2}{\partial t} + \sigma \alpha F_0 \frac{\partial F_1}{\partial x} + \sigma^2 \alpha F_0 \frac{\partial F_2}{\partial x} + \sigma^2 \alpha F_1 \frac{\partial F_1}{\partial x} + \ldots = 0$$

$$\frac{\partial}{\partial t} = \sigma^n \frac{\partial}{\partial \tau} \qquad \frac{\partial}{\partial x} = \sigma^m \frac{\partial}{\partial \xi}$$

"stretched" coordinates

$$\sigma^{n+1} \frac{\partial F_1}{\partial \tau} + \sigma^{n+2} \frac{\partial F_2}{\partial \tau} + \sigma^{m+2} \alpha F_1 \frac{\partial F_1}{\partial \xi} + \sigma \alpha F_0 \left[\sigma^m \frac{\partial F_1}{\partial \xi} + \sigma^{m+1} \frac{\partial F_2}{\partial \xi} \right] = 0$$

$$\sigma^{n+1}\frac{\partial F_1}{\partial \tau} + \sigma^{n+2}\frac{\partial F_2}{\partial \tau} + \sigma^{m+2}\alpha F_1\frac{\partial F_1}{\partial \xi} + \sigma\alpha F_0\left[\sigma^m\frac{\partial F_1}{\partial \xi} + \sigma^{m+1}\frac{\partial F_2}{\partial \xi}\right] = 0$$

$$\left\{\begin{array}{l} n+1 = m+2 \\ F_0 = 0 \end{array}\right. \longrightarrow \begin{array}{l} n = \dfrac{3}{2} \\ m = \dfrac{1}{2} \end{array} \longrightarrow \begin{array}{l} \xi = \sigma^{1/2}x \\ \tau = \sigma^{3/2}t \end{array}$$

$$\sigma^2\left[\frac{\partial F_1}{\partial \tau} + \alpha F_1\frac{\partial F_1}{\partial \xi}\right] + \sigma^3\left[\frac{\partial F_2}{\partial \tau}\right] = 0$$

$$\frac{\partial F_1}{\partial \tau} + \alpha F_1\frac{\partial F_1}{\partial \xi} = 0 \longrightarrow \boxed{\frac{\partial \overline{F}_1}{\partial t} + \alpha\,\overline{F}_1\frac{\partial \overline{F}_1}{\partial x} = 0}$$

nonlinear waves

Nonlinear wave equations

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{(\varepsilon + p)\gamma^2}\left(\vec{\nabla} p + \vec{v}\frac{\partial p}{\partial t}\right)$$ Euler

EOS: $$\vec{\nabla} p = {c_s}^2 \vec{\nabla} \varepsilon$$

If: $\varepsilon \propto \ldots + \ldots \vec{\nabla}^2 \rho_B$ $\longrightarrow$ $\vec{\nabla} p \propto \ldots + \ldots \vec{\nabla}(\vec{\nabla}^2 \rho_B)\ldots$

$$\frac{\partial \rho_1}{\partial t} + \alpha \rho_1 \frac{\partial \rho_1}{\partial x} + \beta \frac{\partial^3 \rho_1}{\partial x^3} = 0$$ Korteweg - de Vries (KdV)

If: $\nabla^2 \ldots = 0$

$$\frac{\partial \rho_1}{\partial t} + \alpha \rho_1 \frac{\partial \rho_1}{\partial x} = 0$$ Breaking wave equation

In real life

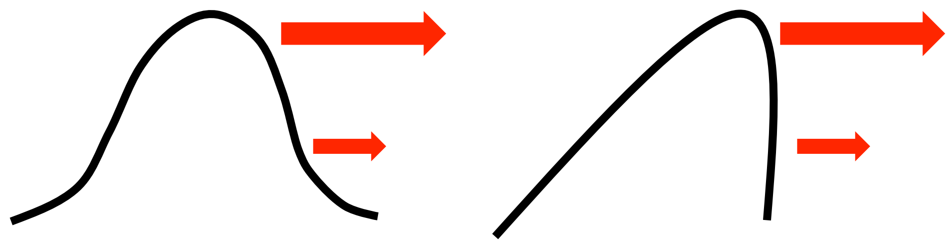
$$\frac{\partial \rho_1}{\partial t} + \alpha \rho_1 \frac{\partial \rho_1}{\partial x} + \beta \frac{\partial^3 \rho_1}{\partial x^3} = 0$$

KdV solitons

$$\frac{\partial \rho_1}{\partial t} + \alpha \rho_1 \frac{\partial \rho_1}{\partial x} = 0$$

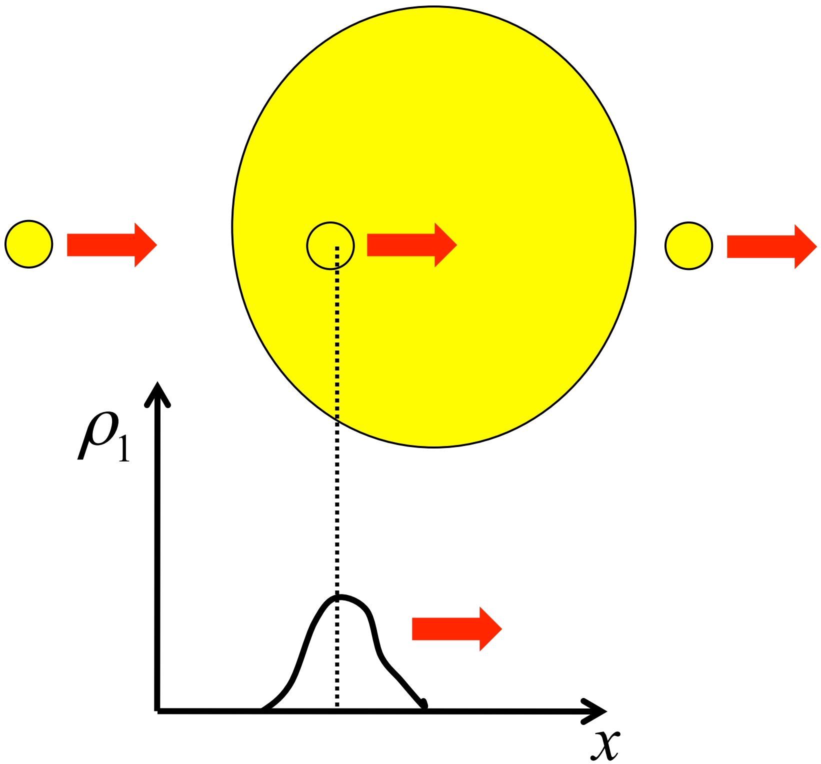
"Breaking wave":

soliton



KdV solitons in cold nuclear matter

$$\hat{\rho}_1(x,t) = \frac{3}{(3-{c_s}^2)}\,\frac{(u-c_s)}{c_s}\;sech^2\Big[\frac{{m_v}^2}{g_v}\sqrt{\frac{(u-c_s)c_sM}{2\rho_0}}(x-ut)\Big]$$



Fowler, Raha, Weiner,
PLB (1982)

Fogaca, Navarra,
PLB (2006) nucl-th/0512097
PLB (2007) nucl-th/0611011

Fogaca, Navarra, Ferreira Filho
NPA (2009) arXiv:0801. 3938

KdV solitons in cold quark matter

$$\frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial x} + \left[\frac{(2 - {c_s}^2)}{2} - \left(\frac{27 g^2\, {\rho_0}^2}{8 {m_G}^2}\right)\frac{(2{c_s}^2 - 1)}{2A} - \frac{\pi^{2/3} {\rho_0}^{4/3}}{A}\left({c_s}^2 - \frac{1}{6}\right)\right] c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial x}$$
$$+ \left[\frac{9 g^2\, {\rho_0}^2 c_s}{8 {m_G}^4 A}\right]\frac{\partial^3 \hat{\rho}_1}{\partial x^3} = 0$$

KdV soliton

$$\left\{\begin{aligned} A &= \left(\frac{27 g^2\, {\rho_0}^2}{8 {m_G}^2}\right) + \pi^{2/3} {\rho_0}^{4/3} \\ {c_s}^2 &= \frac{\left(\frac{27 g^2\, {\rho_0}^2}{8 {m_G}^2}\right) + \pi^{2/3} {\rho_0}^{4/3}}{\left(\frac{27 g^2\, {\rho_0}^2}{8 {m_G}^2}\right) + 3\pi^{2/3} {\rho_0}^{4/3}} \end{aligned}\right.$$

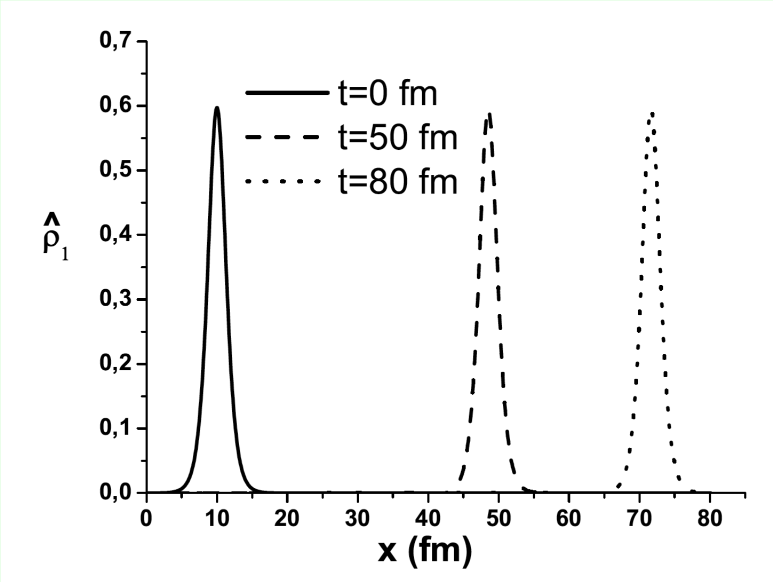
Fogaca, FSN,
PLB (2011) arXiv:1012.5266

Fogaca, FSN, Ferreira Filho
PRD (2011) arXiv:1106.5959

KdV soliton

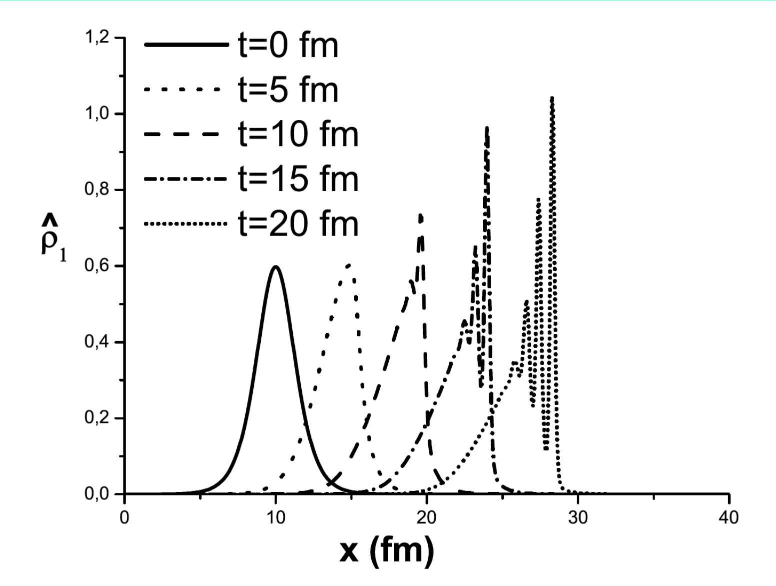
$$\hat{\rho}_1(x,t) = \frac{3(u-c_s)}{\alpha c_s}\ sech^2\left[\sqrt{\frac{(u-c_s)}{4\beta}}(x-ut)\right]$$

$$\lambda = \sqrt{\frac{4\beta}{(u-c_s)}} = \sqrt{\frac{36 g^2\, {\rho_0}^2 c_s}{(u-c_s) {m_G}^4 A}}$$



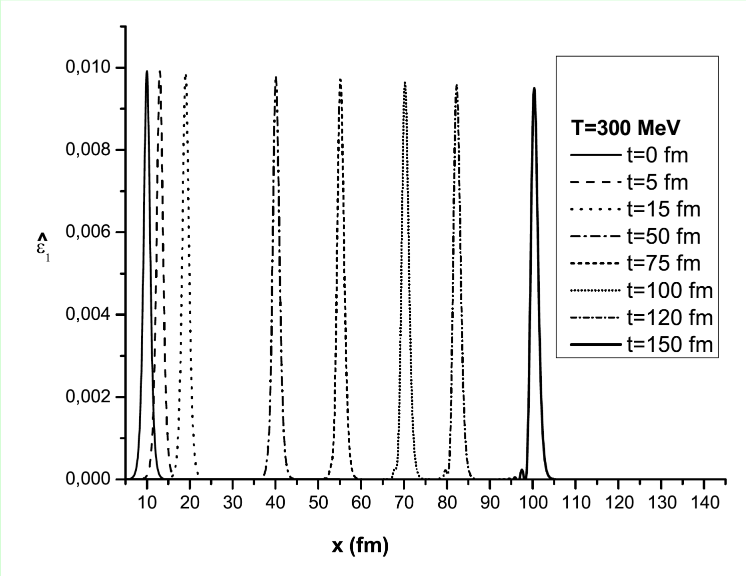
Breaking wave

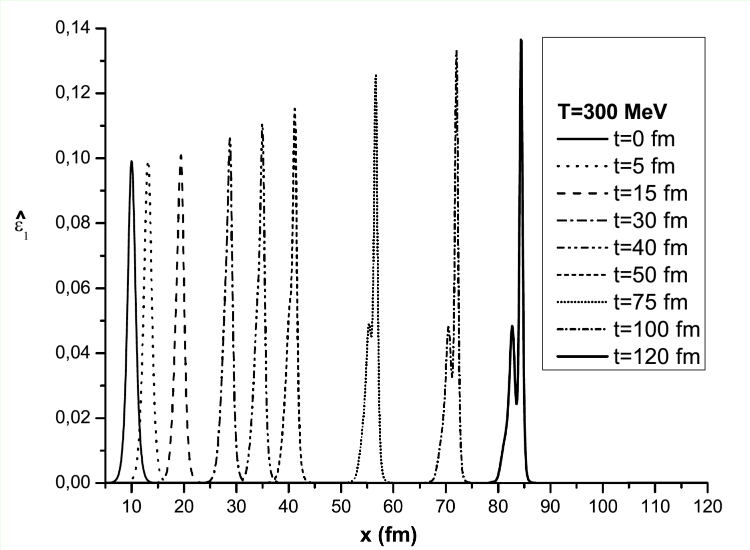
$$\frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial x} + \frac{2}{3} c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial x} = 0$$

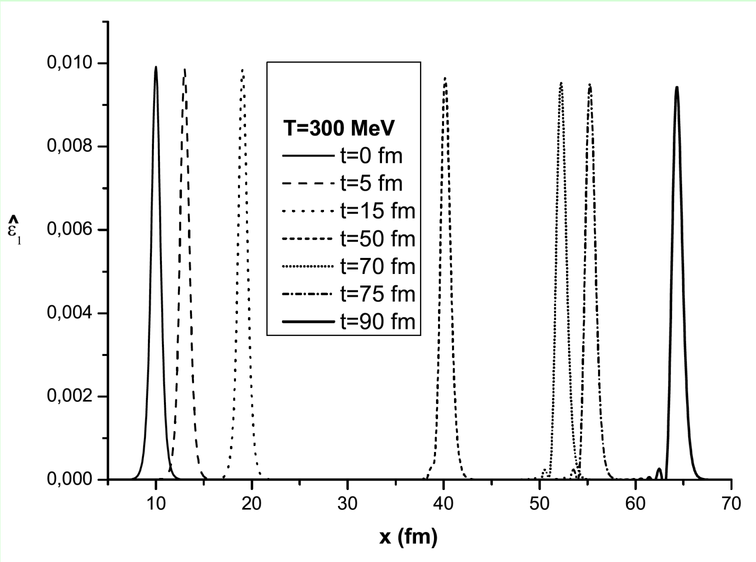


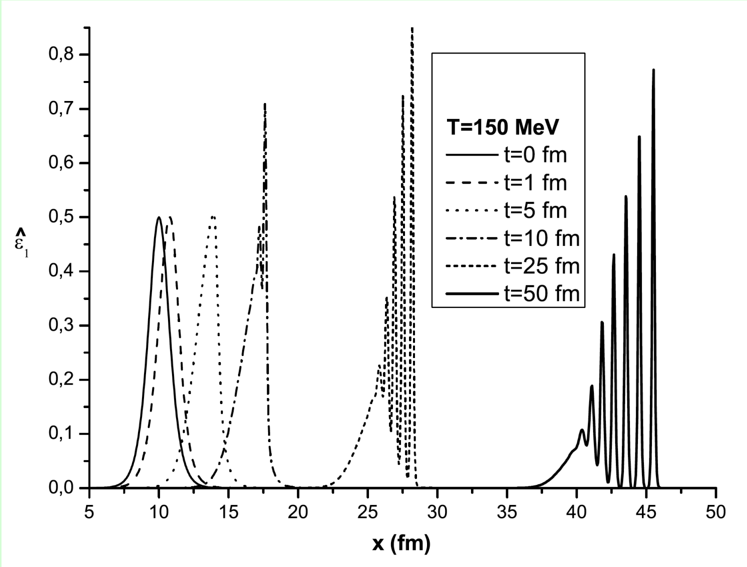
Breaking waves in hot quark matter

$$\frac{\partial \hat{\varepsilon}_1}{\partial t} + c_S \frac{\partial \hat{\varepsilon}_1}{\partial x} + \left[1 + \frac{1}{3}\left(\frac{T_c}{T_0}\right)^4\right] \frac{c_S}{2} \hat{\varepsilon}_1 \frac{\partial \hat{\varepsilon}_1}{\partial x} = 0$$







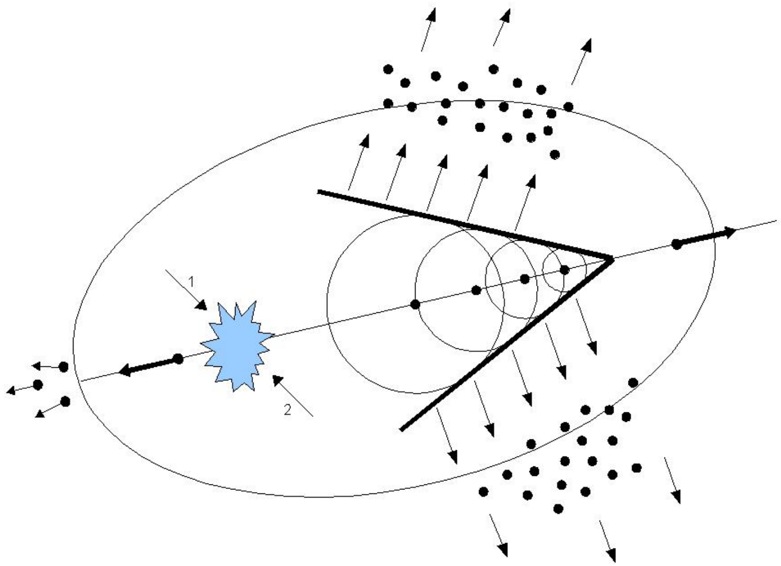


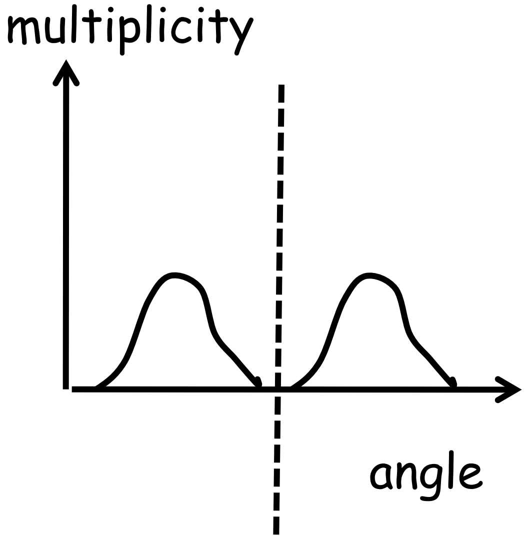
Localized structures live for a long time

How to observe this thing ?

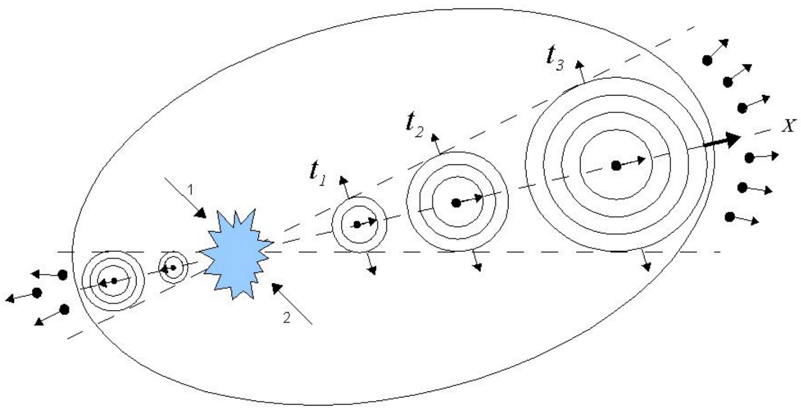
Localized structures live for a long time

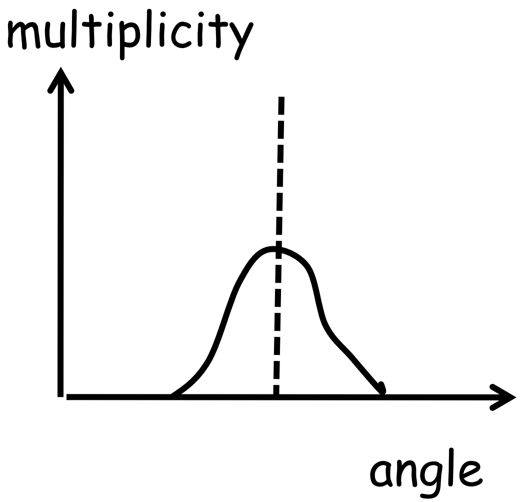
Mach cone





soliton like behavior





“sound ball” ?

Summary

Analytical studies complement numerical simulations

There may be not-so-small perturbations in QGP

We need to go beyond linearization (RPM)

Nonlinear equations: KdV, Breaking wave, Burgers...

KdV solitons may exist in hadronic matter and in QGP

Localized structures live long and might be observable

Relativistic Navier-Stokes leads to the Burger's equation

Fogaca, FSN, Ferreira Filho
NPA (2012) arXiv:1201.0943

Takeshi:

thanks for (30 years of) your attention !

Back ups

$$\frac{\partial \mathcal{L}}{\partial \eta_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \eta_i)} + \partial_\nu \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \eta_i)} \right] = 0$$

$$\mathcal{B} = \frac{37}{90} \pi^2 (T_c)^4$$

$m_G \rightarrow \infty$

$$\frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial x} + \frac{2}{3} c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial x} = 0$$

Breaking wave

$$T^{\mu\nu} = \frac{\partial \mathcal{L}^*}{\partial(\partial_\mu \eta_i)}(\partial^\nu \eta_i) - g^{\mu\nu}\mathcal{L}^* - \left[\partial_\beta \frac{\partial \mathcal{L}^*}{\partial(\partial_\mu \partial_\beta \eta_i)}\right](\partial^\nu \eta_i) + \frac{\partial \mathcal{L}^*}{\partial(\partial_\mu \partial_\beta \eta_i)}(\partial_\beta \partial^\nu \eta_i)$$

$$\longrightarrow \quad \varepsilon = < T_{00} > \qquad p = \frac{1}{3} < T_{ii} >$$

$$\left\{ \begin{array}{l} \nabla^2 V_0 = 0 \quad \longrightarrow \quad V_0 = \frac{g_V}{{m_V}^2}\rho_B \\[2em] V_0 = \frac{g_V}{{m_V}^2}\rho_B \quad \longrightarrow \quad \nabla^2 V_0 = \frac{g_V}{m_V^2}\nabla^2 \rho_B \quad \longrightarrow \quad -\vec{\nabla}^2 V_0 + {m_V}^2 V_0 = g_V \bar{\psi}\gamma^0\psi \end{array} \right.$$

$$\Downarrow$$

$$V_0 = \frac{g_V}{{m_V}^2}\rho_B + \frac{g_V}{{m_V}^4}\vec{\nabla}^2 \rho_B$$

$$\left\{ \; \varepsilon = \frac{{g_V}^2}{2{m_V}^2}{\rho_B}^2 + \frac{{g_V}^2}{2{m_V}^4}\rho_B \vec{\nabla}^2 \rho_B + \ldots \right.$$

Nonlinear wave equations

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{(\varepsilon+p)\gamma^2}\left(\vec{\nabla}p + \vec{v}\frac{\partial p}{\partial t}\right)$$ Euler

EOS: $p = {c_s}^2 \varepsilon \longrightarrow \vec{\nabla}p = {c_s}^2\vec{\nabla}\varepsilon$

If: $\varepsilon \propto ... + ...\vec{\nabla}^2\rho_B \longrightarrow \vec{\nabla}p \propto ... + ...\vec{\nabla}(\vec{\nabla}^2\rho_B)...$

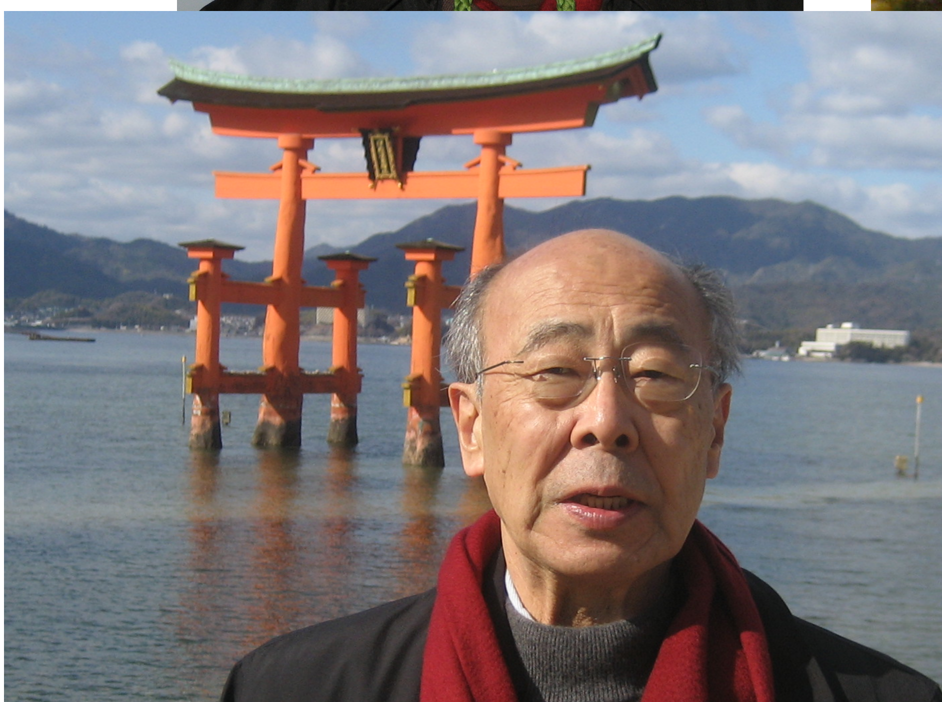
$$\frac{\partial \rho_1}{\partial t} + \alpha\rho_1\frac{\partial \rho_1}{\partial x} + \beta\frac{\partial^3 \rho_1}{\partial x^3} = 0$$ Korteweg - de Vries (KdV)

If: $\nabla^2 ... = 0$

$$\frac{\partial \rho_1}{\partial t} + \alpha\rho_1\frac{\partial \rho_1}{\partial x} = 0$$ Breaking wave equation





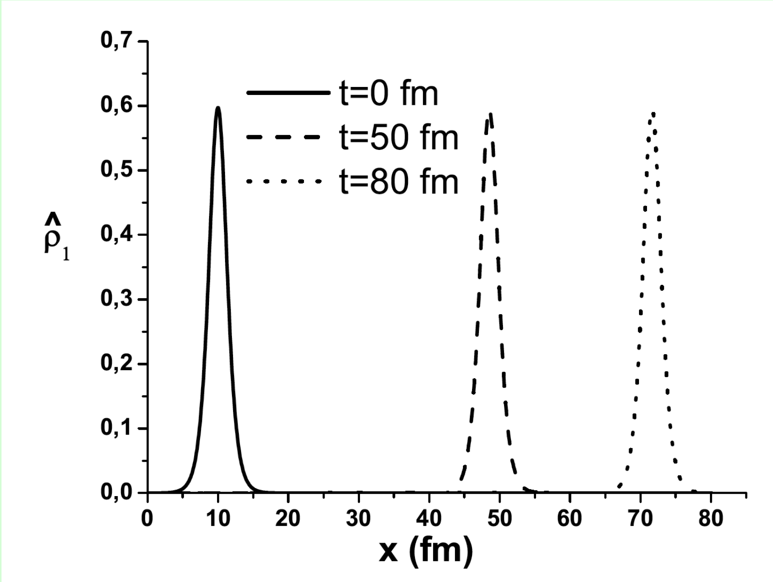




KdV soliton

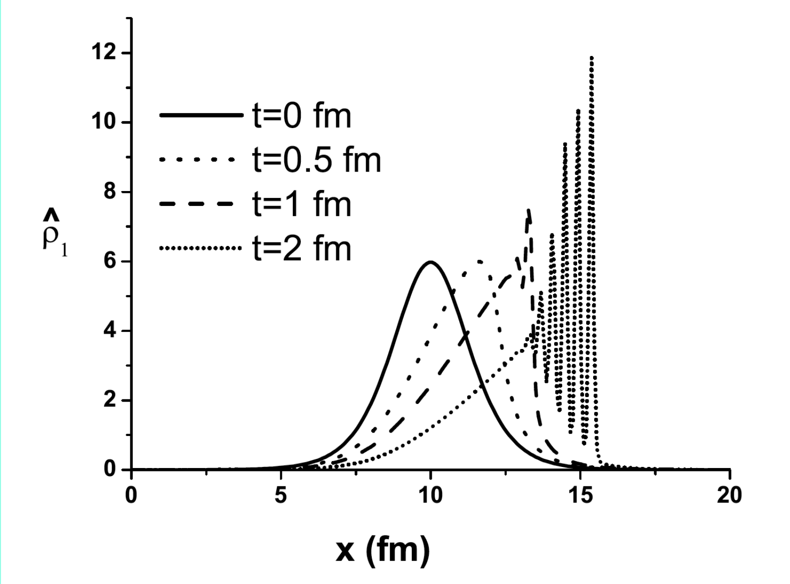
$$\hat{\rho}_1(x,t) = \frac{3(u-c_s)}{\alpha c_s}\; sech^2\left[\sqrt{\frac{(u-c_s)}{4\beta}}(x-ut)\right]$$

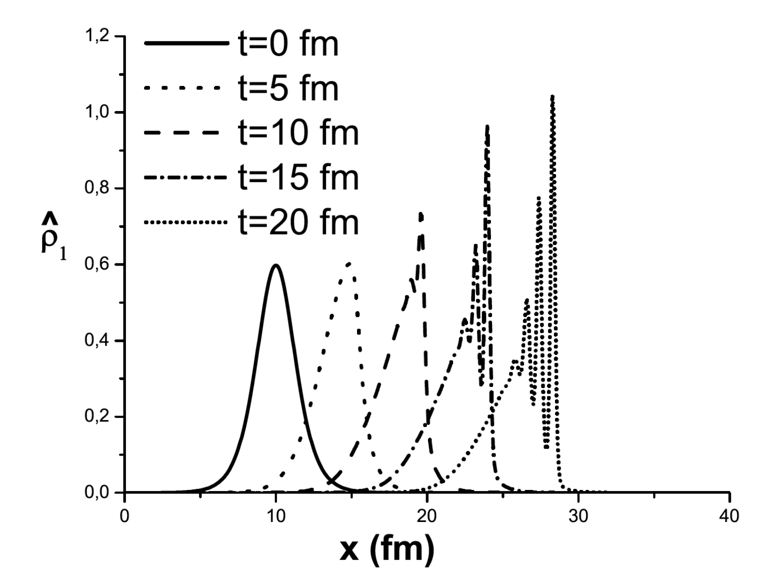
$$\lambda = \sqrt{\frac{4\beta}{(u-c_s)}} = \sqrt{\frac{36 g^2 \rho_0{}^2 c_s}{(u-c_s) m_G{}^4 A}}$$



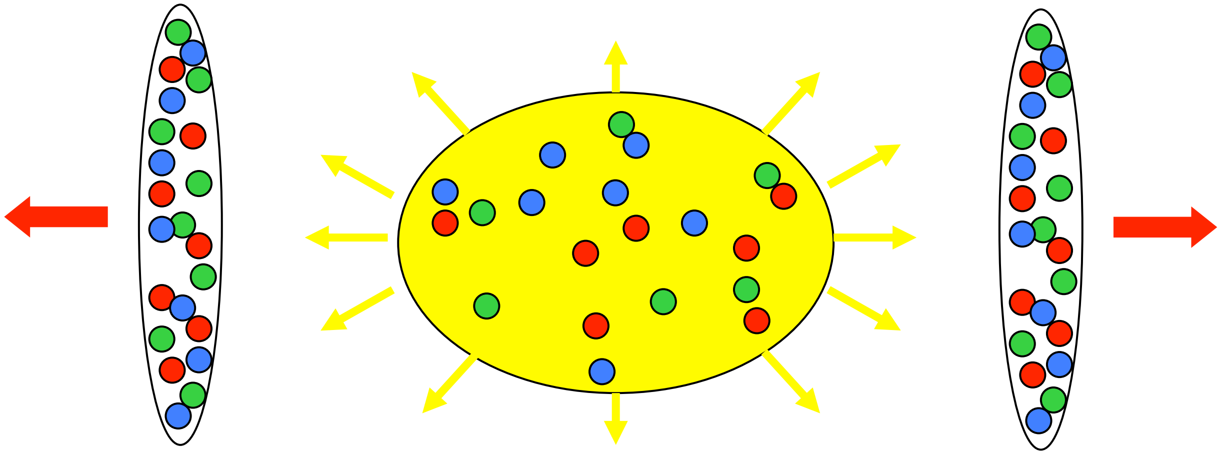
Breaking wave

$$\frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial x} + \frac{2}{3} c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial x} = 0$$





The Quark Gluon Plasma



2004: Discovery of the "perfect fluid"

(Elliptic flow and jet quenching)

"The first act of hydrodynamics": the fluid

2005: Evidence of Mach cones in the QGP

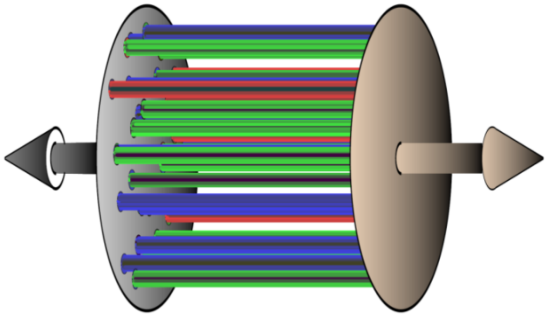
("Double bump" structure in the "away side jets")

"The second act of hydrodynamics": waves in the fluid

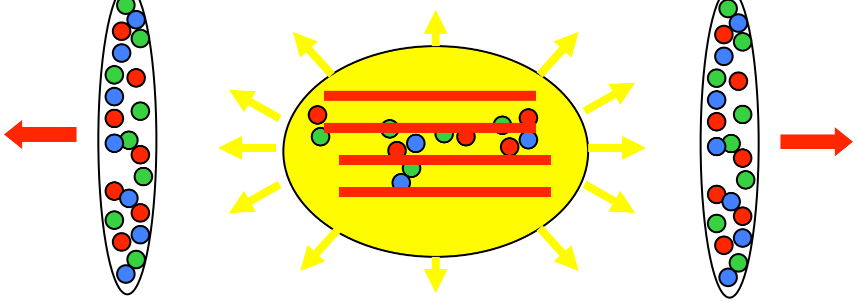
E. Shuryak

Perturbations and waves in the QGP

Perturbations from the initial conditions



“Glasma” (CGC) generates
“flux tubes”

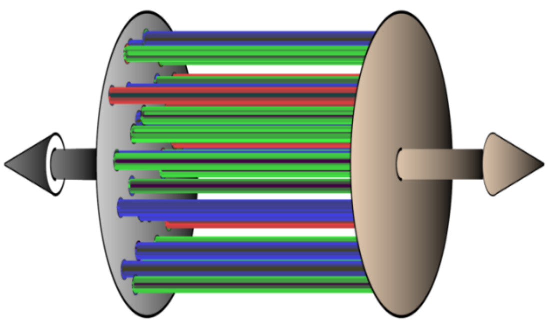


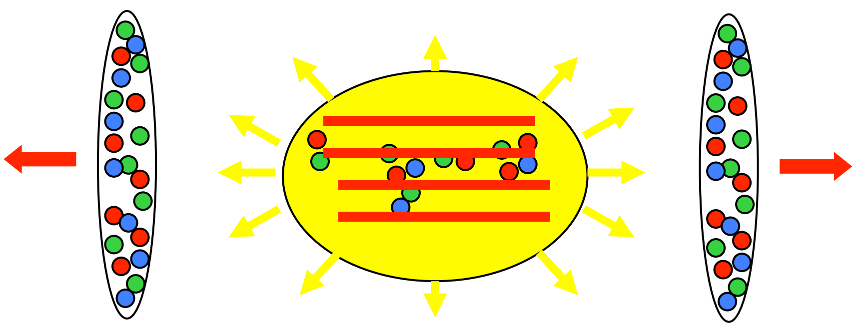
Fireball + flux tubes ?

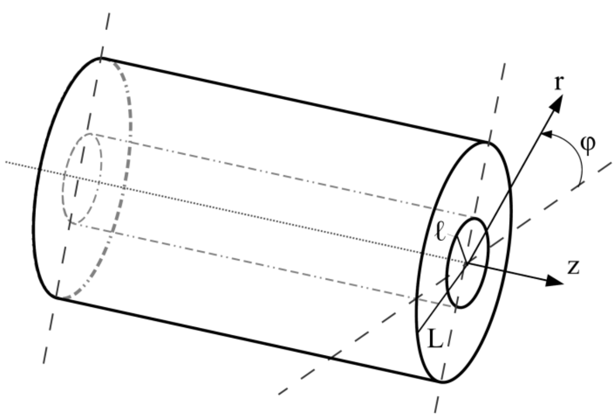
Expansion of tubes in viscous hydrodynamics

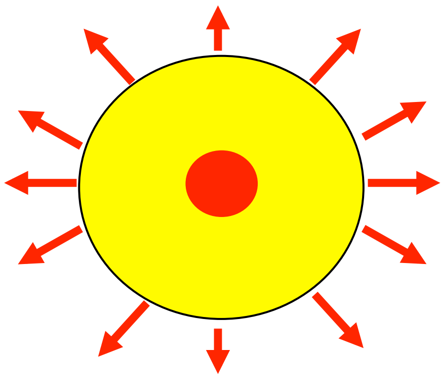
Fogaça, Navarra, Ferreira Filho,
Nucl. Phys. A887 (2012) 22

Non-linear perturbations in the relativistic Navier - Stokes equation









Relativistic Viscous Hydrodynamics

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{(\varepsilon + p)\gamma^2}\left(\vec{\nabla}p + \vec{v}\frac{\partial p}{\partial t}\right)$$

Perfect

$$\begin{aligned}
&(\varepsilon + p)\gamma^2\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\vec{v} + \vec{v}\frac{\partial p}{\partial t} + \vec{\nabla}p \\
&-\eta\vec{v}\left\{\partial_\mu\partial^\mu\gamma + \partial_\mu\frac{\partial u^\mu}{\partial t} - \partial_\mu\left[\gamma\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)(\gamma u^\mu)\right]\right\} \\
&-\vec{v}\left(\zeta - \frac{2}{3}\eta\right)\frac{\partial}{\partial t}\left[\frac{\partial\gamma}{\partial t} + \vec{\nabla}\cdot(\gamma\vec{v})\right] + \vec{v}\left(\zeta - \frac{2}{3}\eta\right)\partial_\mu\left\{\gamma u^\mu\left[\frac{\partial\gamma}{\partial t} + \vec{\nabla}\cdot(\gamma\vec{v})\right]\right\} \\
&+\eta\left\{\partial_\mu\partial^\mu(\gamma\vec{v}) - \partial_\mu\vec{\nabla}u^\mu - \partial_\mu\left[\gamma\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)(\gamma\vec{v}u^\mu)\right]\right\} \\
&-\left(\zeta - \frac{2}{3}\eta\right)\vec{\nabla}\left[\frac{\partial\gamma}{\partial t} + \vec{\nabla}\cdot(\gamma\vec{v})\right] - \left(\zeta - \frac{2}{3}\eta\right)\partial_\mu\left\{\gamma\vec{v}u^\mu\left[\frac{\partial\gamma}{\partial t} + \vec{\nabla}\cdot(\gamma\vec{v})\right]\right\} = 0
\end{aligned}$$

Viscous

Relativistic Navier-Stokes equation

Entropy

$$\frac{\partial s}{\partial t} + \gamma^2 v s\left(\frac{\partial v}{\partial t} + \vec{v}\cdot\vec{\nabla} v\right) + \vec{\nabla}\cdot(s\vec{v}) = 0$$

Perfect

$$\gamma\frac{\partial s}{\partial t} + \gamma\vec{\nabla}s\cdot\vec{v} + s\frac{\partial\gamma}{\partial t} + s\vec{\nabla}\gamma\cdot\vec{v} + \gamma s\vec{\nabla}\cdot\vec{v}$$
$$= -\frac{\eta}{T}\left(\frac{\partial\gamma}{\partial t}\right)^2 - 2\frac{\eta}{T}\left[\vec{\nabla}\gamma\cdot\frac{\partial}{\partial t}(\gamma\vec{v})\right] - \frac{\eta}{T}(\partial^i u^j)\partial_j u_i$$
$$+ \frac{1}{T}\left(\frac{2}{3}\eta + \zeta\right)\left[\frac{\partial\gamma}{\partial t} + \gamma\vec{\nabla}\cdot\vec{v} + \vec{\nabla}\gamma\cdot\vec{v}\right]^2$$

Viscous

η = shear viscosity

ζ = bulk viscosity

Equation of State: MIT Bag Model

EOS

$$\begin{cases} \varepsilon = \mathcal{B} + \dfrac{\gamma_G}{(2\pi)^3}\displaystyle\int d^3k\ \ k\ \ (e^{k/T}-1)^{-1} + \dfrac{\gamma_Q}{(2\pi)^3}\displaystyle\int d^3k\ \ k\ \ [n_{\vec{k}} + \bar{n}_{\vec{k}}] \\ p = -\mathcal{B} + \dfrac{1}{3}\left\{\dfrac{\gamma_G}{(2\pi)^3}\displaystyle\int d^3k\ \ k\ \ (e^{k/T}-1)^{-1} + \dfrac{\gamma_Q}{(2\pi)^3}\displaystyle\int d^3k\ k\left[n_{\vec{k}} + \bar{n}_{\vec{k}}\right]\right\} \end{cases}$$

$$c_S{}^2 = \frac{\partial p}{\partial \varepsilon} = \frac{1}{3}$$

$$\varepsilon = \frac{37}{30}\pi^2\left(T^4 + T_B^4\right) \qquad \mathcal{B} = \frac{37}{30}\pi^2 (T_B)^4$$

$$s = 4\frac{37}{90}\pi^2 T^3$$

$$s = 4\frac{37}{90}\pi^2\left[\frac{30}{37\pi^2}(\varepsilon - \mathcal{B})\right]^{3/4}$$

The wave equation

Define dimensionless variables:

$$\hat{\varepsilon} = \frac{\varepsilon}{\varepsilon_0} \qquad \hat{v} = \frac{v}{c_s} \qquad \hat{v}_r = \frac{v_r}{c_s}, \qquad \hat{v}_z = \frac{v_z}{c_s}$$

Introduce "stretched" coordinates

$$R = \frac{\sigma^{1/2}}{L}(r - c_s t), \qquad Z = \frac{\sigma}{L} z, \qquad T = \frac{\sigma^{3/2}}{L} c_s t$$

Expand Euler and Continuity equations around equilibrium:

$$\begin{cases} \hat{\varepsilon} = 1 + \sigma \varepsilon_1 + \sigma^2 \varepsilon_2 + \sigma^3 \varepsilon_3 + \cdots \\ \hat{v}_r = \sigma v_{r_1} + \sigma^2 v_{r_2} + \sigma^3 v_{r_3} + \cdots \\ \hat{v}_z = \sigma^{3/2} v_{z_1} + \sigma^{5/2} v_{z_2} + \sigma^{7/2} v_{z_3} + \cdots \end{cases}$$

Obtain non-linear wave equations and go back to original coordinates

$$\frac{\partial}{\partial r}\left\{\frac{\partial\hat{\varepsilon}_1}{\partial t}+c_s\frac{\partial\hat{\varepsilon}_1}{\partial r}+\frac{c_s}{2}\left[1+\left(\frac{T_B}{T_0}\right)^4\right]\hat{\varepsilon}_1\frac{\partial\hat{\varepsilon}_1}{\partial r}+\frac{\hat{\varepsilon}_1}{2t}\right\}+\frac{c_s}{2}\frac{\partial^2\hat{\varepsilon}_1}{\partial z^2}=0$$

perfect

$$\frac{\partial}{\partial r}\left\{\frac{\partial\hat{\varepsilon}_1}{\partial t}+c_s\frac{\partial\hat{\varepsilon}_1}{\partial r}+\frac{c_s}{2}\left[1+\left(\frac{T_B}{T_0}\right)^4\right]\hat{\varepsilon}_1\frac{\partial\hat{\varepsilon}_1}{\partial r}+\frac{\hat{\varepsilon}_1}{2t}\right.$$
$$\left.-\frac{1}{T_0}\left(\frac{\zeta}{s}+\frac{4}{3}\frac{\eta}{s}\right)\frac{\partial^2\hat{\varepsilon}_1}{\partial r^2}\right\}+\frac{c_s}{2}\frac{\partial^2\hat{\varepsilon}_1}{\partial z^2}=0$$

viscous

$$\hat{\varepsilon}_1 \equiv \sigma\varepsilon_1$$

Background expansion: Bjorken cooling

$$\frac{T(\tau)}{T(\tau_0)}=\left(\frac{\tau_0}{\tau}\right)^{1/3} \quad \Longrightarrow \quad \frac{T_B}{T_0}\to\frac{T_B}{T_0(\tau)}=\frac{T_B}{T_0(\tau_0)}\left(\frac{\tau}{\tau_0}\right)^{1/3}$$

$$\tau=\sqrt{t^2-r^2}$$

Numerical solutions

$$\frac{c_s}{2}\frac{\partial^2\hat{\varepsilon}_1}{\partial z^2}=0$$

longitudinal invariance

$$\frac{\partial\hat{\varepsilon}_1}{\partial t}+c_s\frac{\partial\hat{\varepsilon}_1}{\partial r}+\frac{c_s}{2}\left[1+\left(\frac{T_B}{T_0}\right)^4\right]\hat{\varepsilon}_1\frac{\partial\hat{\varepsilon}_1}{\partial r}+\frac{\hat{\varepsilon}_1}{2t}=0$$

perfect

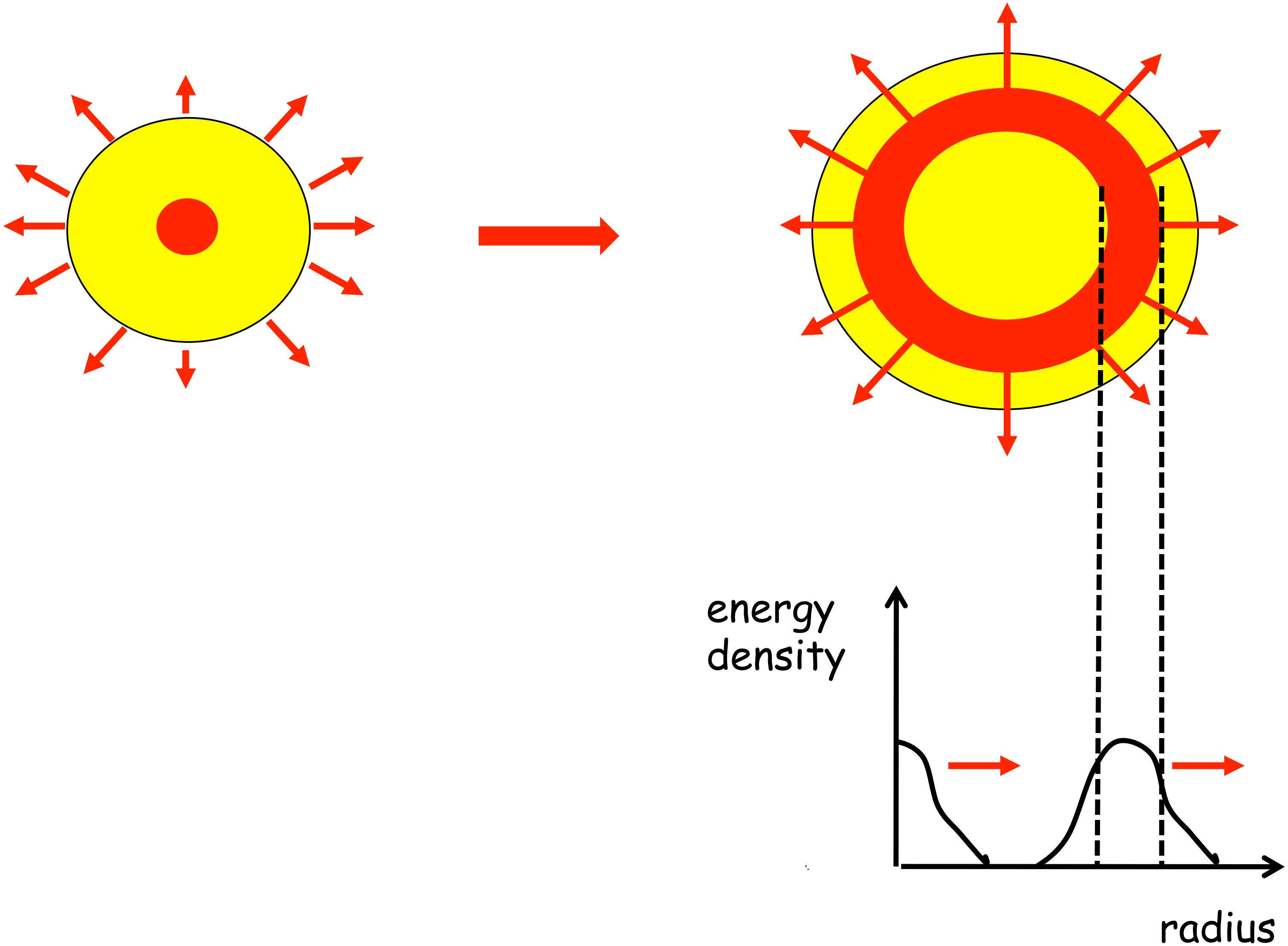
$$\frac{\partial\hat{\varepsilon}_1}{\partial t}+c_s\frac{\partial\hat{\varepsilon}_1}{\partial r}+\frac{c_s}{2}\left[1+\left(\frac{T_B}{T_0}\right)^4\right]\hat{\varepsilon}_1\frac{\partial\hat{\varepsilon}_1}{\partial r}+\frac{\hat{\varepsilon}_1}{2t}=\frac{1}{T_0}\left(\frac{\zeta}{s}+\frac{4}{3}\frac{\eta}{s}\right)\frac{\partial^2\hat{\varepsilon}_1}{\partial r^2}$$

viscous

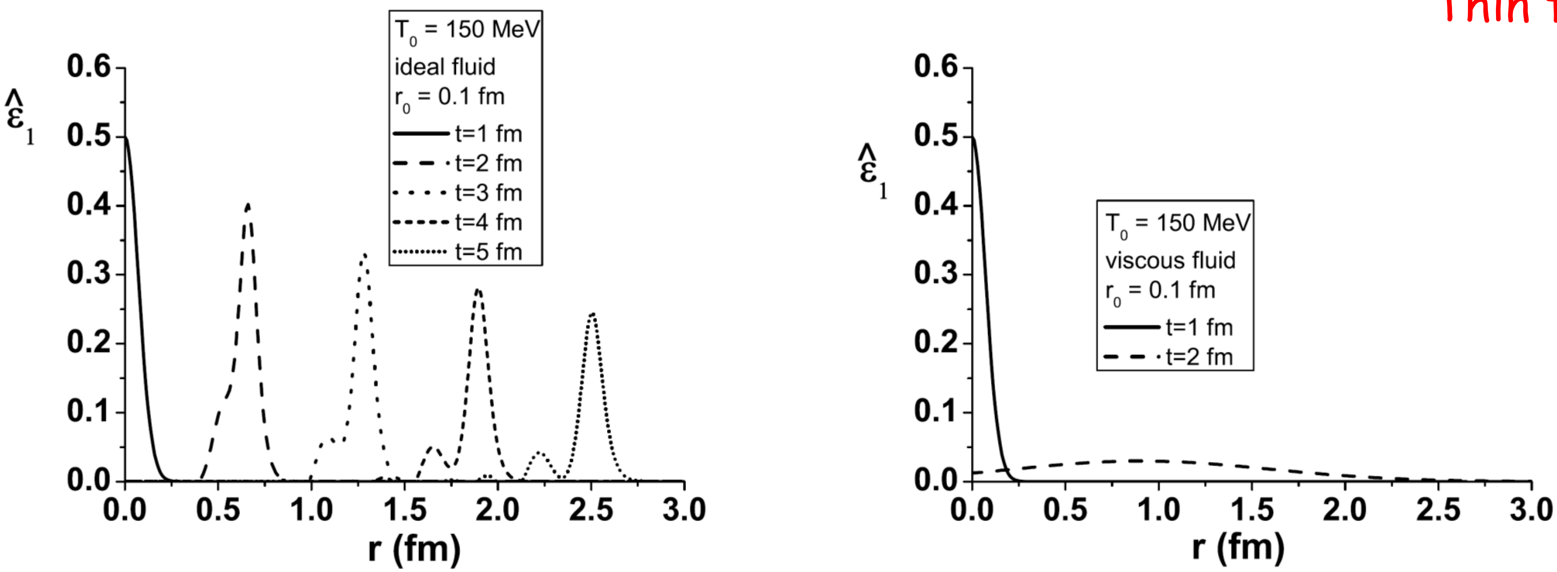
Burgers equation

Initial condition $\hat{\varepsilon}_1=Ae^{-r^2/r_0^2}$ $\qquad \dfrac{\eta}{s}=0.16$

"Tube" becomes a "ring"



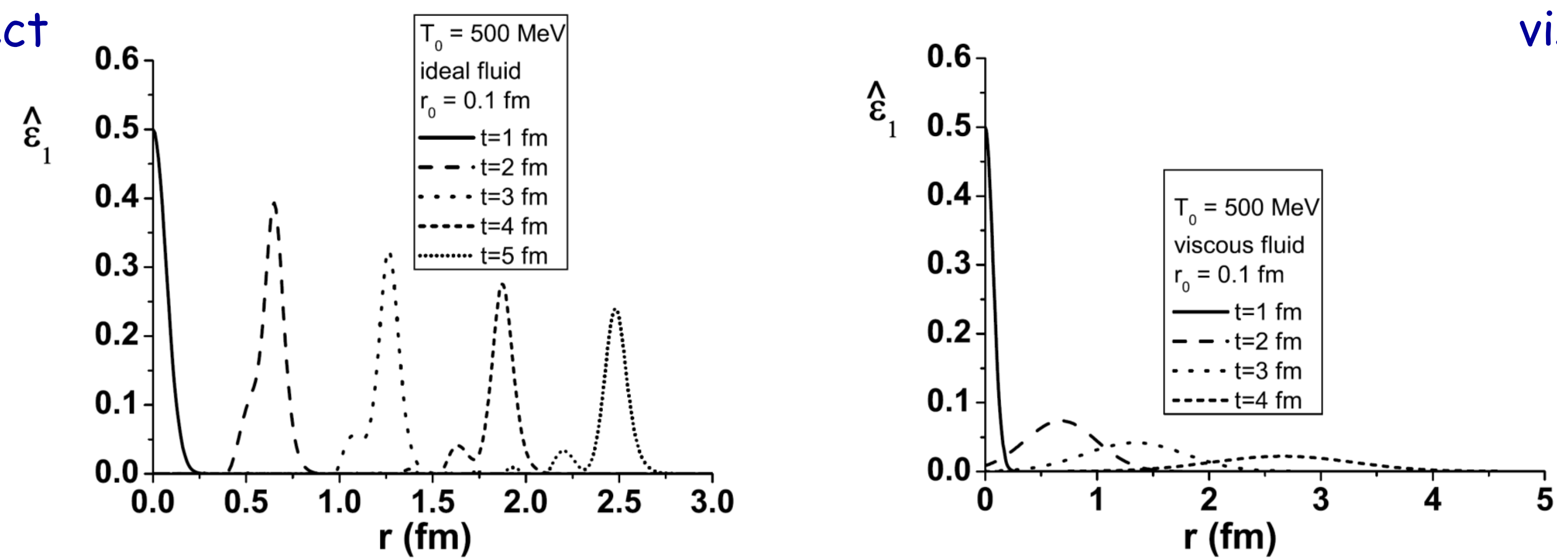
“Thin tube”



(a) (b)

perfect

viscous



Fogaça, Navarra, Ferreira Filho,
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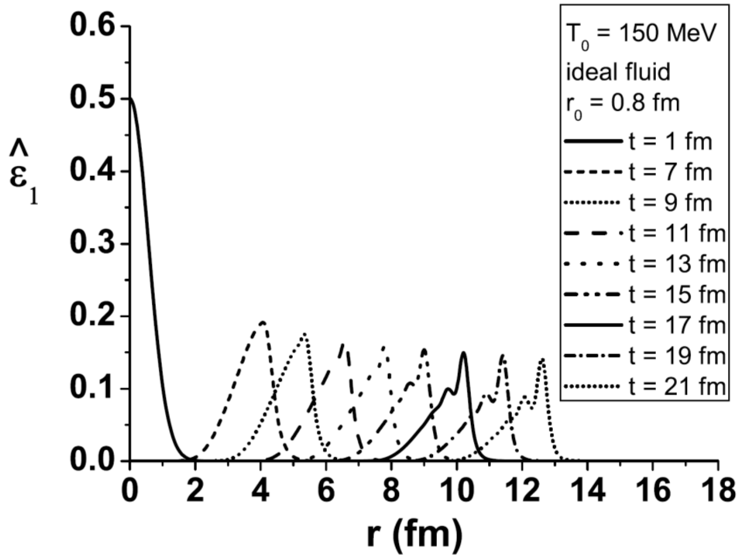
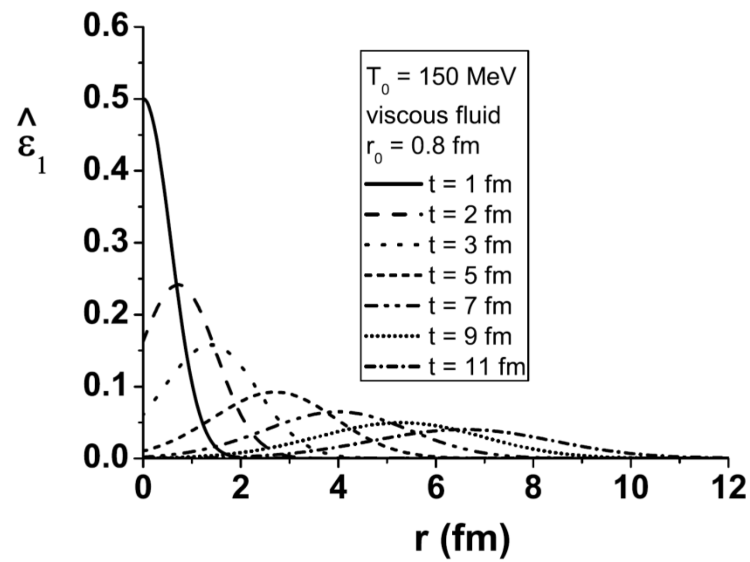
“Thick tube”



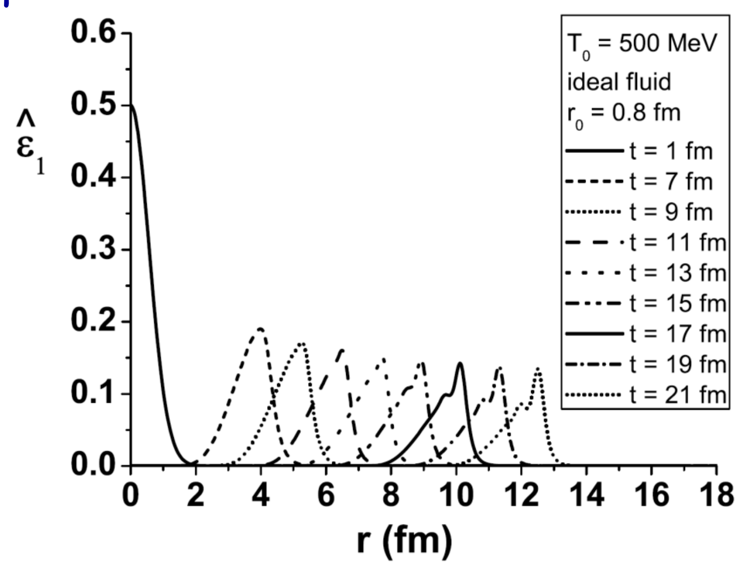
(a)

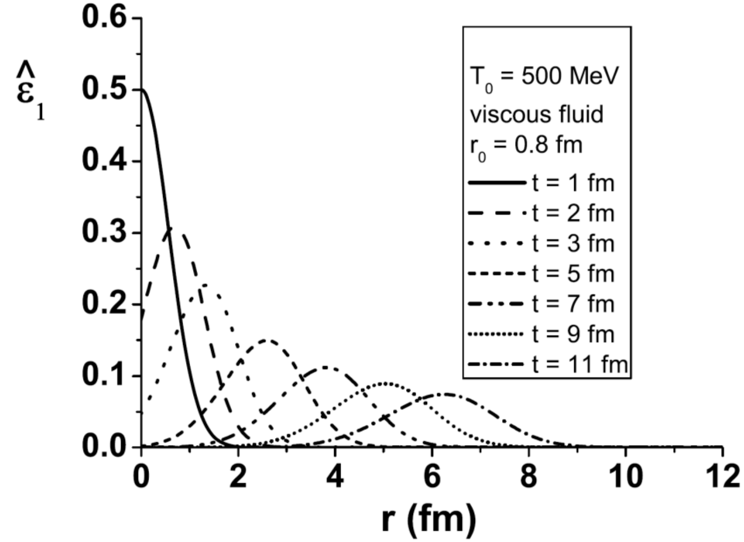


(b)

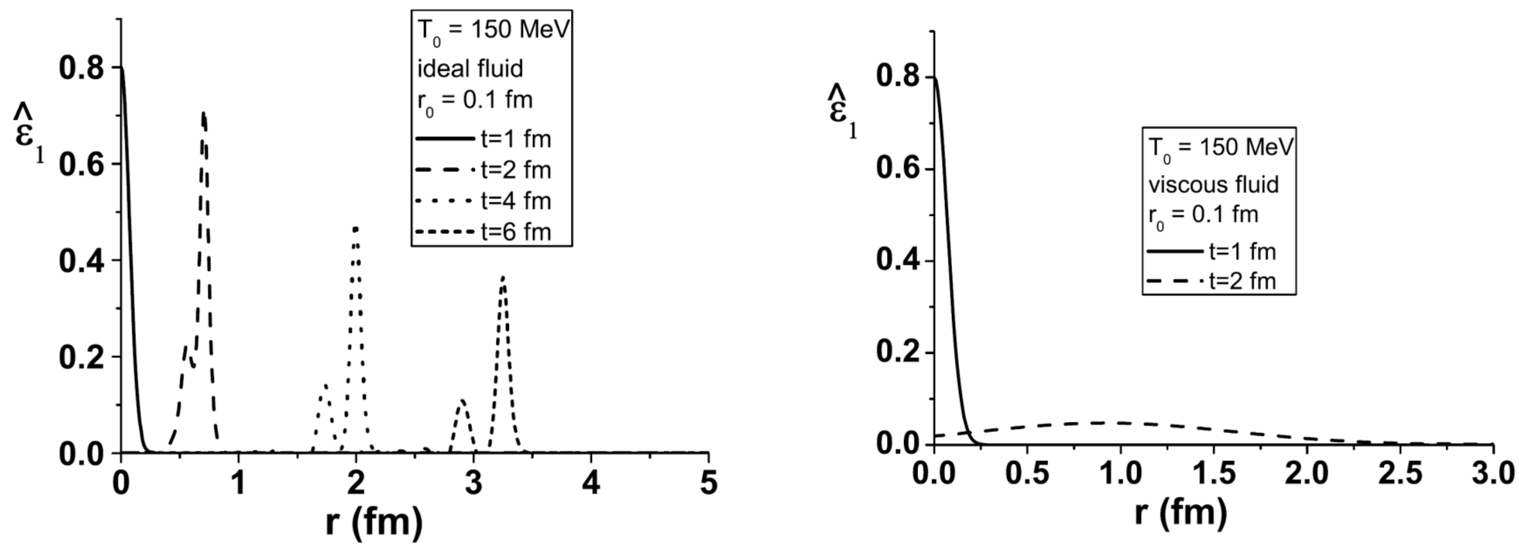
perfect

viscous





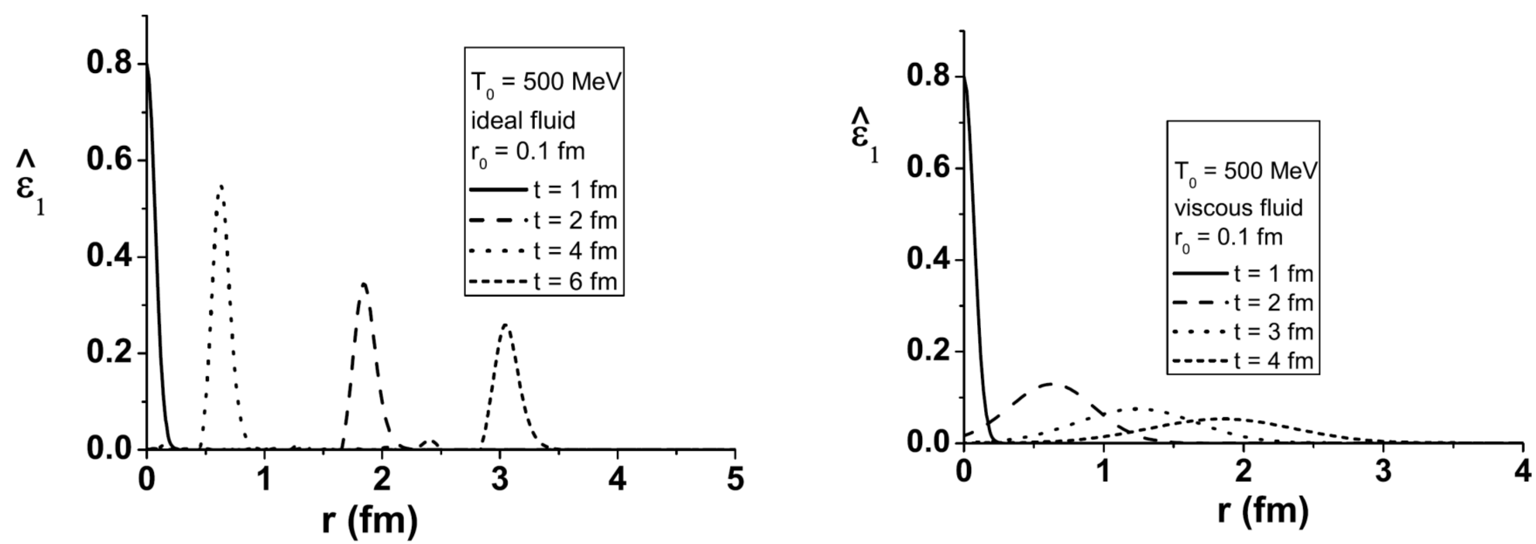
“Thin” and
more energetic



(a) (b)

perfect

viscous



Estimating the Laplacian :

$$V_0 = \frac{g_V}{{m_V}^2}\rho_B \Rightarrow \nabla^2 V_0 = \frac{g_V}{m_V^2}\nabla^2\rho_B \Rightarrow -\vec{\nabla}^2 V_0 + {m_V}^2 V_0 = g_V\bar{\psi}\gamma^0\psi$$

$$\Downarrow$$

$$V_0 = \frac{g_V}{{m_V}^2}\rho_B + \frac{g_V}{{m_V}^4}\vec{\nabla}^2\rho_B \Leftarrow {m_V}^2 V_0 = \frac{g_V}{{m_V}^2}\vec{\nabla}^2\rho_B + g_V\rho_B$$

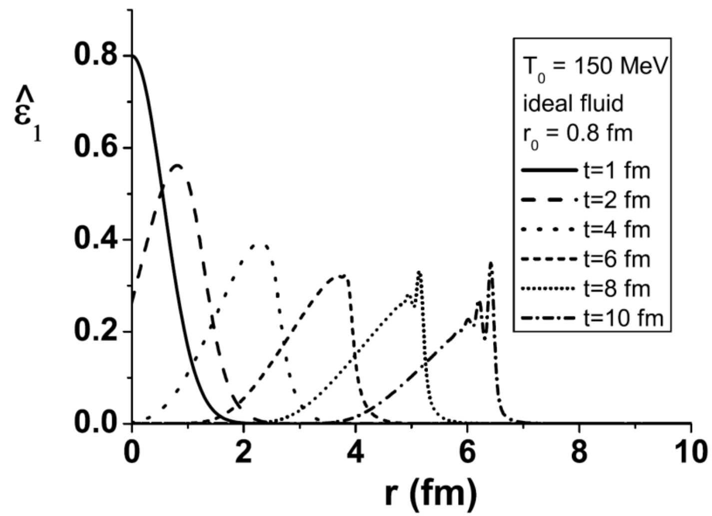
We calculate the energy momentum tensor and the equation of state

$$\varepsilon = \frac{1}{2}\left\{\frac{\partial}{\partial t}\left[\frac{(M-M^*)}{g_S}\right]\right\}^2 + \frac{1}{2}\left\{\vec{\nabla}\left[\frac{(M-M^*)}{g_S}\right]\right\}^2 + \frac{{m_S}^2}{2{g_S}^2}(M-M^*)^2 +$$

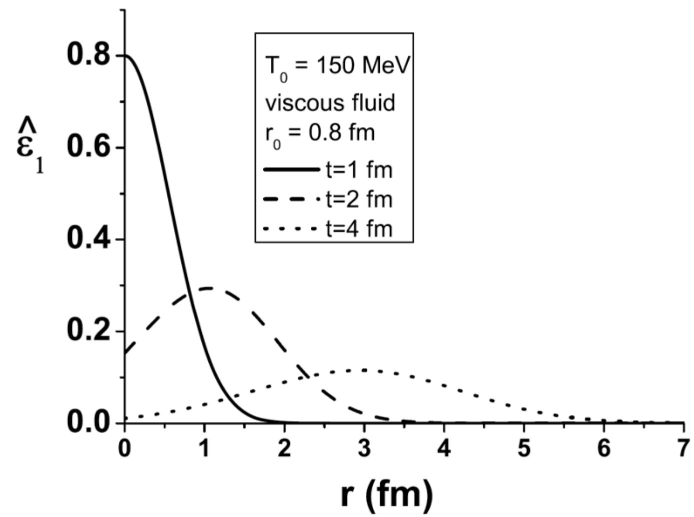
$$+b\frac{(M-M^*)^3}{3{g_S}^3} + c\frac{(M-M^*)^4}{4{g_S}^4} + \frac{{g_V}^2}{2{m_V}^2}{\rho_B}^2 + \frac{{g_V}^2}{2{m_V}^4}\rho_B\vec{\nabla}^2\rho_B +$$

$$+\frac{\gamma_s}{(2\pi)^3}\int_0^{k_F} d^3k(\vec{k}^2 + {M^*}^2)^{1/2}$$

"Thick" and more energetic



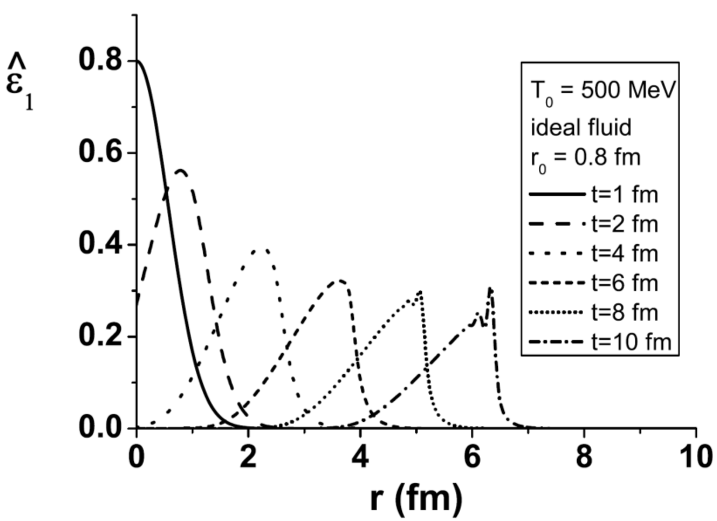
(a)

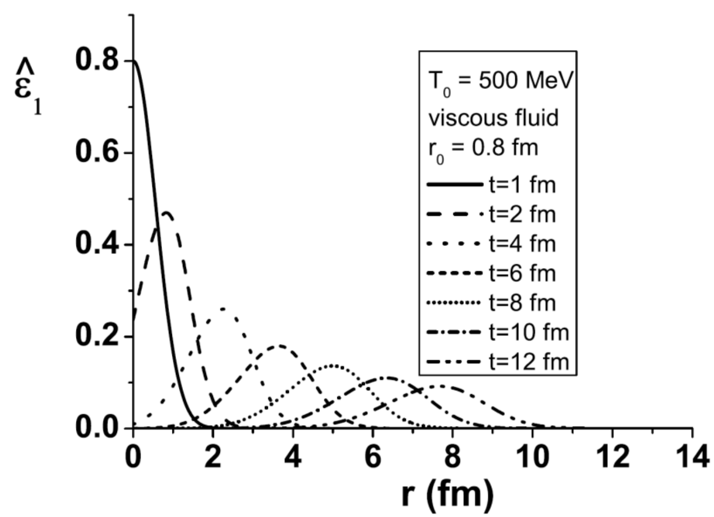


(b)

perfect

viscous





What is important for the evolution of the tubes

Viscosity: spreads the energy faster and damps the tube

Initial radius: larger tubes live longer

Initial amplitude: higher tubes become unstable sooner

Background temperature

Background expansion

Analytical solution

Full 3-d problem: $\hat{\varepsilon}_1 = \hat{\varepsilon}_1\ (r, \varphi, z, t)$

$$\frac{\partial}{\partial r}\left\{\frac{\partial \hat{\varepsilon}_1}{\partial t} + c_s \frac{\partial \hat{\varepsilon}_1}{\partial r} + \alpha \hat{\varepsilon}_1 \frac{\partial \hat{\varepsilon}_1}{\partial r} - \nu \frac{\partial^2 \hat{\varepsilon}_1}{\partial r^2} + \frac{\hat{\varepsilon}_1}{2t}\right\} + \frac{1}{2c_s t^2}\frac{\partial^2 \hat{\varepsilon}_1}{\partial \varphi^2} + \frac{c_s}{2}\frac{\partial^2 \hat{\varepsilon}_1}{\partial z^2} = 0$$

$$\alpha \equiv \frac{c_s}{2}\left[1 + \left(\frac{T_B}{T_0}\right)^4\right] \qquad \nu \equiv \frac{1}{2T_0}\left(\frac{\zeta}{s} + \frac{4}{3}\frac{\eta}{s}\right)$$

$$\hat{\varepsilon}_1(r, z, \varphi, t) = \frac{2\delta A}{c_s T_0}\left(\frac{\zeta}{s} + \frac{4}{3}\frac{\eta}{s}\right)\left[1 + \left(\frac{T_B}{T_0}\right)^4\right]^{-1}$$
$$-\frac{2\delta A}{c_s T_0}\left(\frac{\zeta}{s} + \frac{4}{3}\frac{\eta}{s}\right)\left[1 + \left(\frac{T_B}{T_0}\right)^4\right]^{-1} \times$$
$$\times\, tanh\left\{\delta\left[Ar + Bz - A\frac{c_s \varphi^2 t}{2} - \left(Ac_s + \frac{B^2 c_s}{2A} + \frac{\delta A^2}{T_0}\left(\frac{\zeta}{s} + \frac{4}{3}\frac{\eta}{s}\right)\right)t\right]\right\}$$

$$\frac{\partial F}{\partial t} + \alpha F \frac{\partial F}{\partial x} = 0$$

$$F(x,t) = F_0 + \sigma F_1(x,t) + \sigma^2 F_2(x,t) + \sigma^3 F_3(x,t) + \ldots$$

$$\sigma \frac{\partial F_1}{\partial t} + \sigma^2 \frac{\partial F_2}{\partial t} + \sigma \alpha F_0 \frac{\partial F_1}{\partial x} + \sigma^2 \alpha F_0 \frac{\partial F_2}{\partial x} + \sigma^2 \alpha F_1 \frac{\partial F_1}{\partial x} + \ldots = 0$$

$$\frac{\partial}{\partial t} = \sigma^n \frac{\partial}{\partial \tau} \qquad \frac{\partial}{\partial x} = \sigma^m \frac{\partial}{\partial \xi}$$

$$\sigma^{n+1} \frac{\partial F_1}{\partial \tau} + \sigma^{n+2} \frac{\partial F_2}{\partial \tau} + \sigma^{m+2} \alpha F_1 \frac{\partial F_1}{\partial \xi} + \sigma \alpha F_0 \left[\sigma^m \frac{\partial F_1}{\partial \xi} + \sigma^{m+1} \frac{\partial F_2}{\partial \xi} \right] = 0$$

$$n + 1 = m + 2 \qquad n + 2 \geq 3$$

$$\xi = \sigma^{1/2} x$$

$$m \vec{a} = \vec{F} \qquad \frac{\partial F_1}{\partial \tau} + \alpha F_1 \frac{\partial F_1}{\partial \xi} = 0$$

$$\tau = \sigma^{3/2} t$$

H. Leblond, J. Phys. B: At. Mol. Opt. Phys. **41**, 043001 (2008).

[16] H. Washimi and T. Taniuti, Phys. Rev. Lett. **17**, 996 (1966).

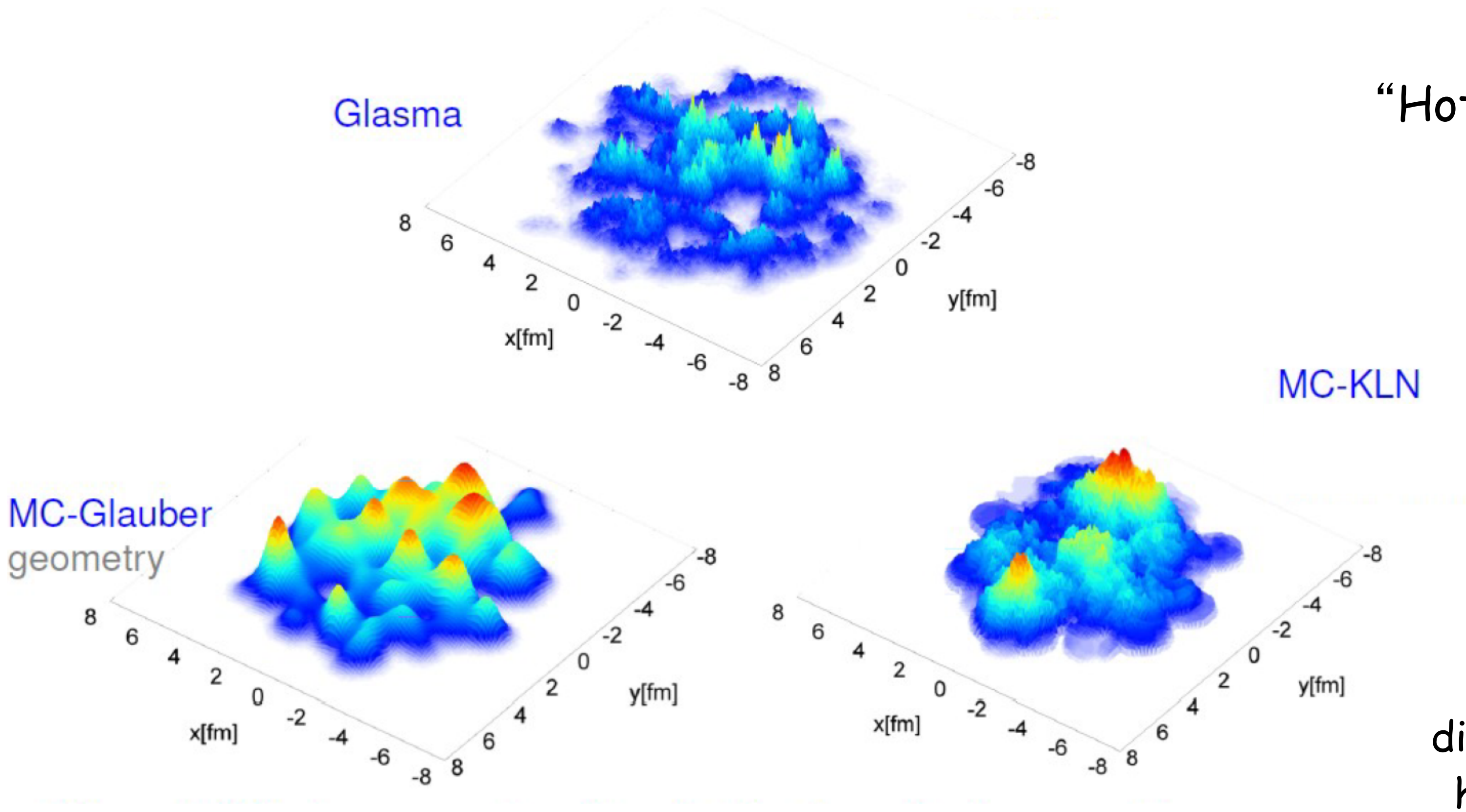
[17] R.C. Davidson, "Methods in Nonlinear Plasma Theory", Academic Press, New York an London, (1972).

[18] For a recent review and a historical account see: H. Leblond, J. Phys. B: At. Mol. Opt. Phys. **41**, 043001 (2008).

[19] Lokenath Debnath, "Nonlinear Partial Differential Equations for Scientists and Engineers", third edition, Birkhäuser, USA, (2011).

Initial energy density distribution in the transverse plane

“Hot spots”



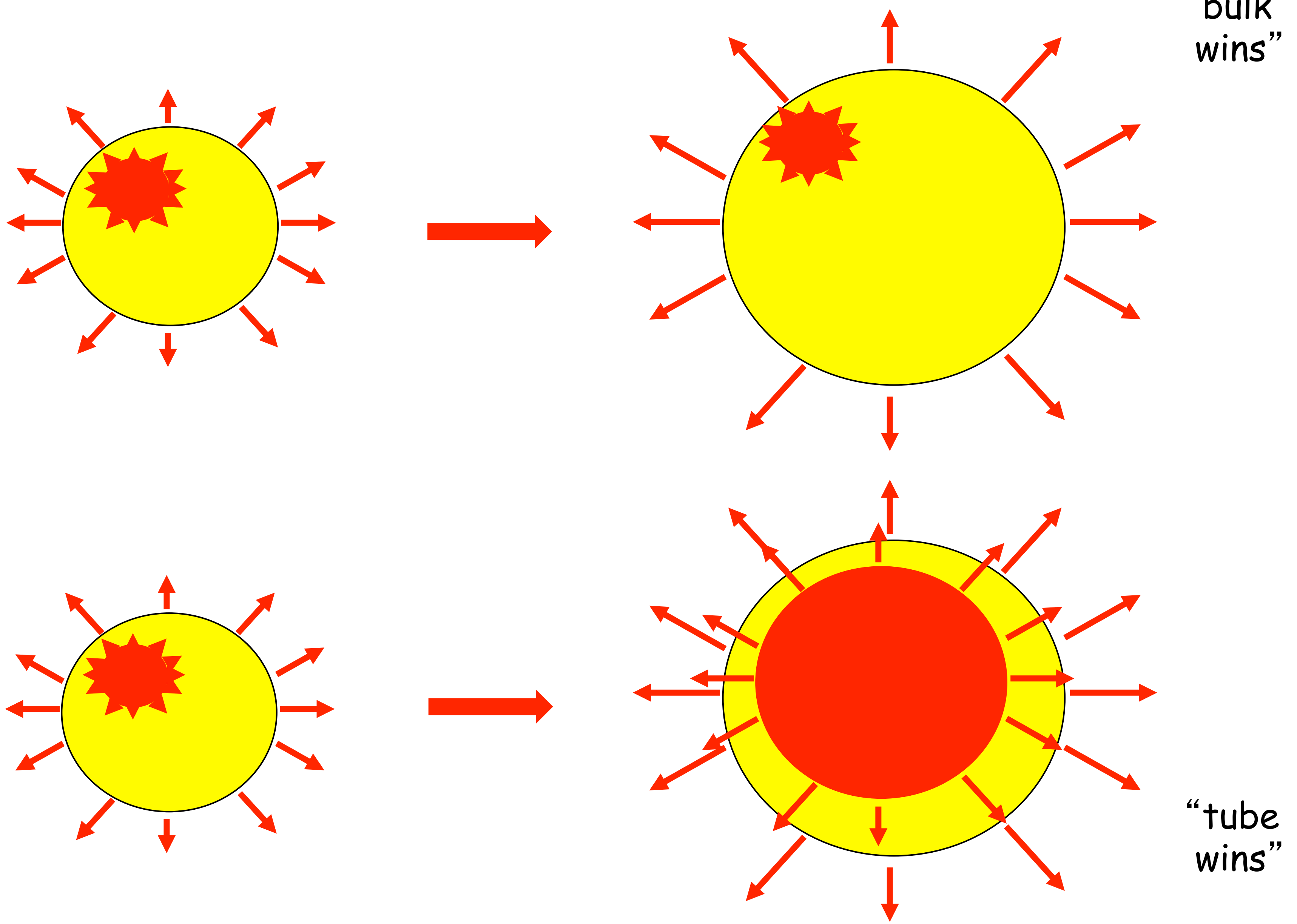
different

heights

widths

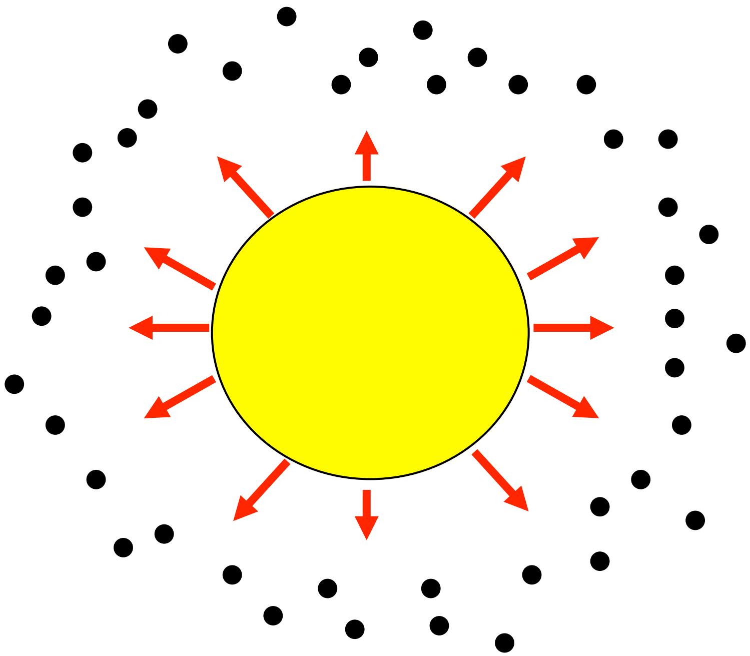
Do these structures survive the expansion ?

"Fight" between tube and bulk

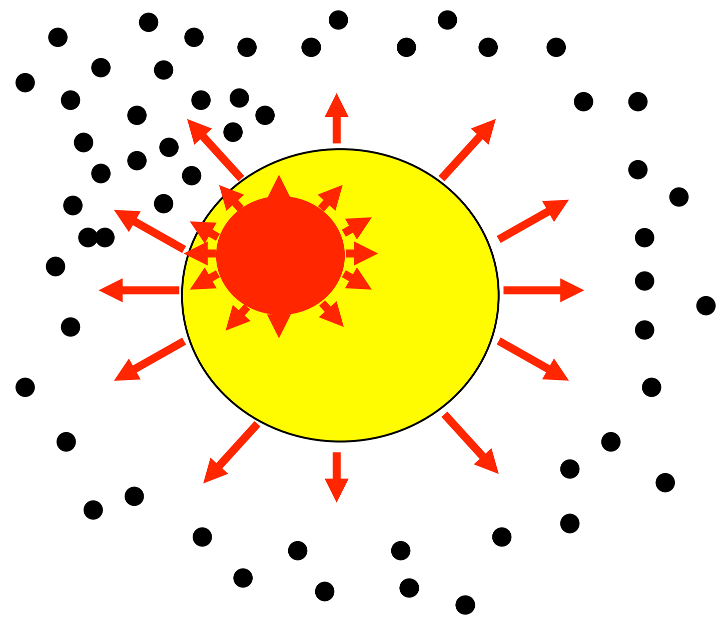


Perturbations caused by flux tubes

Expanding and hadronizing QGP: transverse view



isotropic
distribution

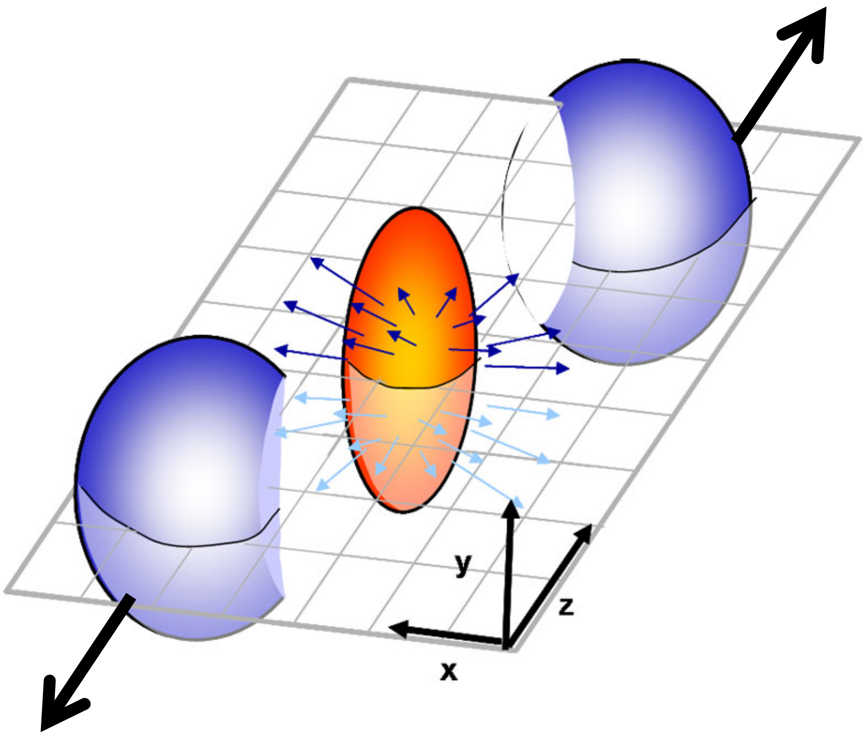


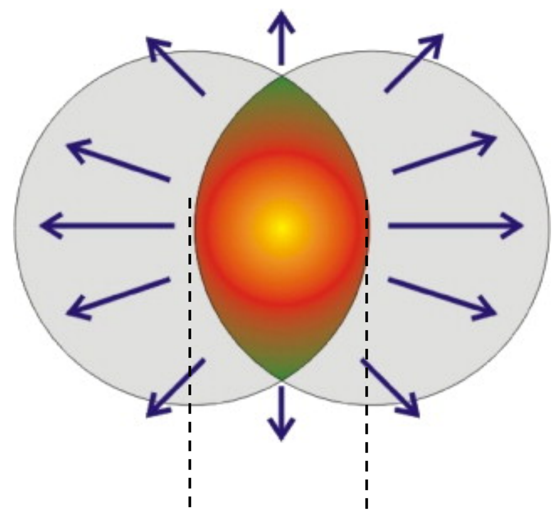
anisotropic
distribution

How to study the evolution of these tubes ?

Elliptic flow

Elliptic flow (v2)





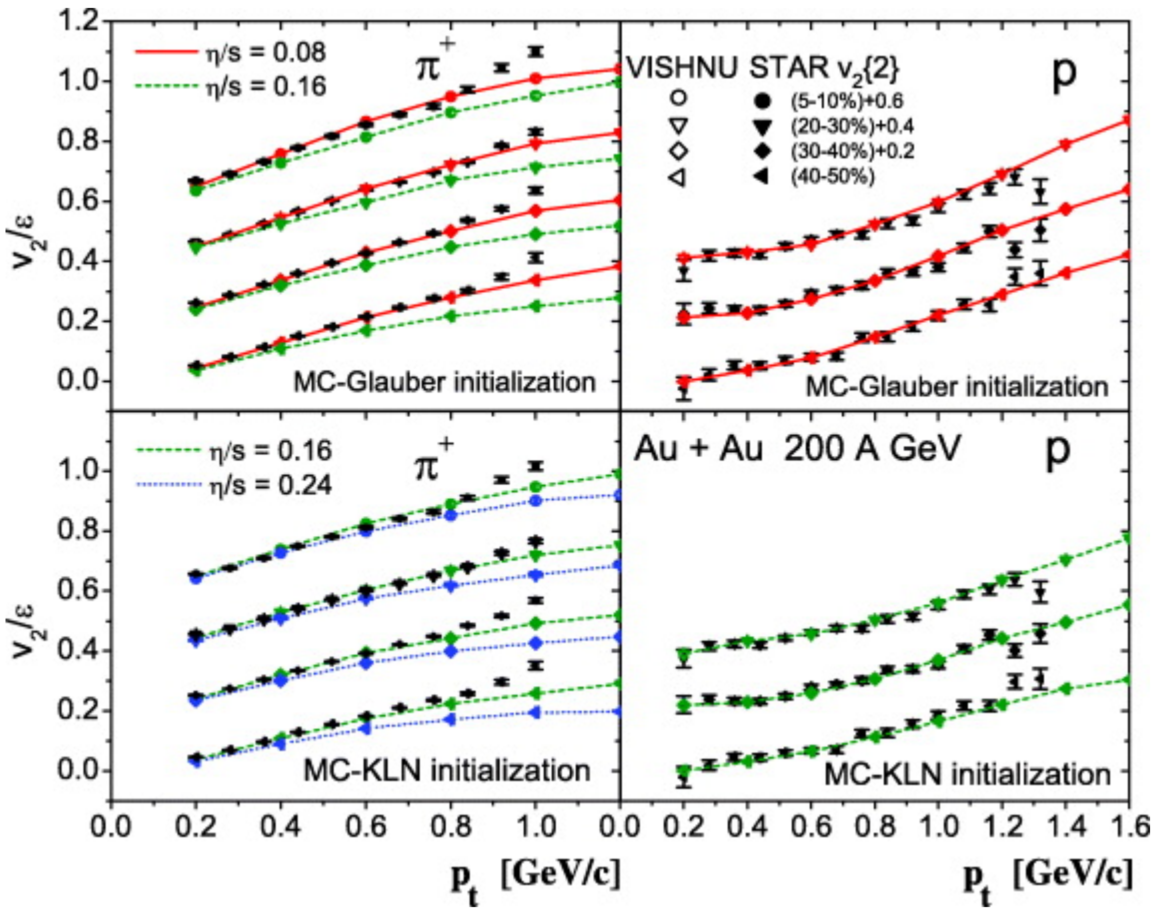
$$\frac{\Delta p}{\Delta x} > \frac{\Delta p}{\Delta y}$$

$$m\,\vec{a} = \vec{F} \quad \longrightarrow \quad \rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\,.\,\vec{\nabla})\,\vec{v}\right] = -\vec{\nabla}\,p \qquad \text{(Euler)}$$

Different pressure gradients deform the transverse momentum distribution

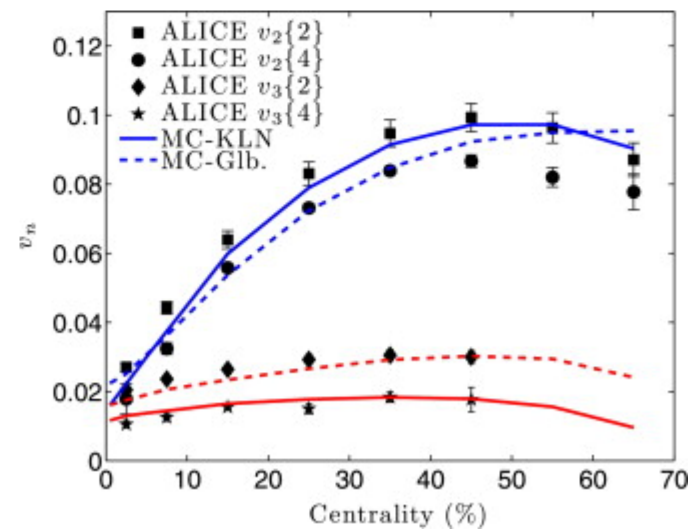
Hydrodynamics describes the data !

RHIC



Snellings, NJP (2011)

LHC

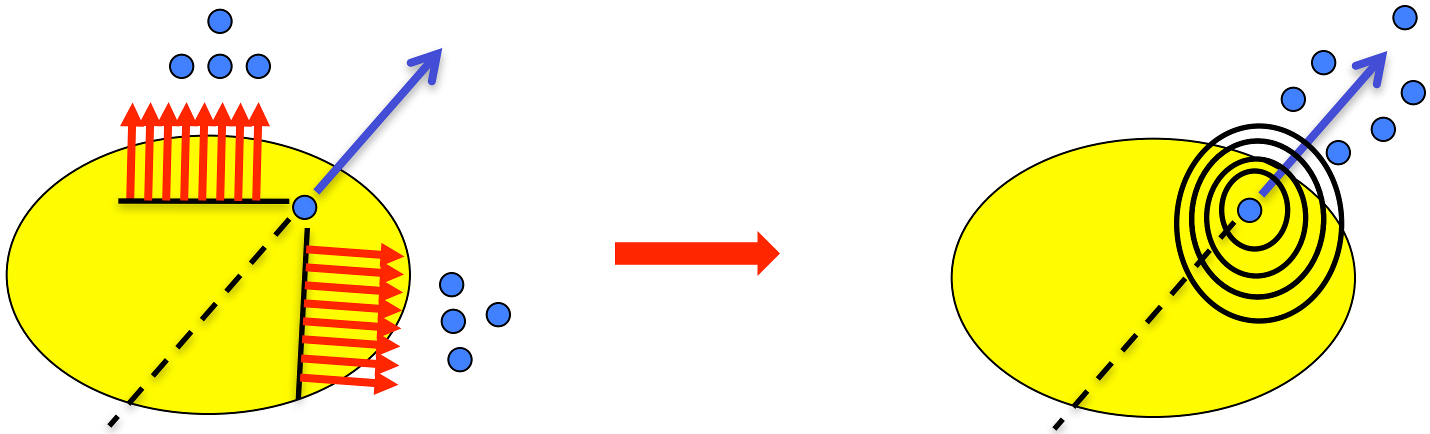


Qiu,Shen,Heinz, PLB (2012)

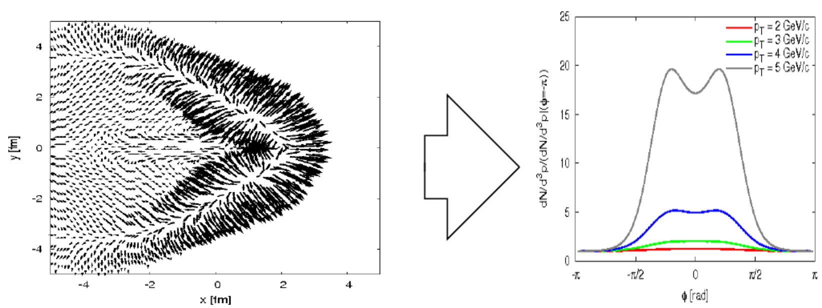
Now what ?

Mach cones

If the pulse lives longer more energy goes foreward
and less energy goes to the conical wave !

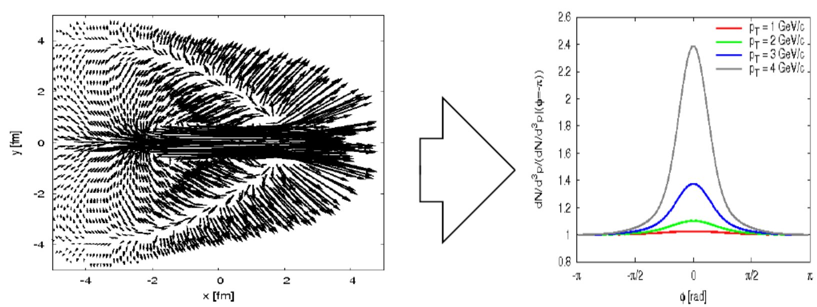


weaker wake



Torrieri, Noronha, Betz, Gyulassy

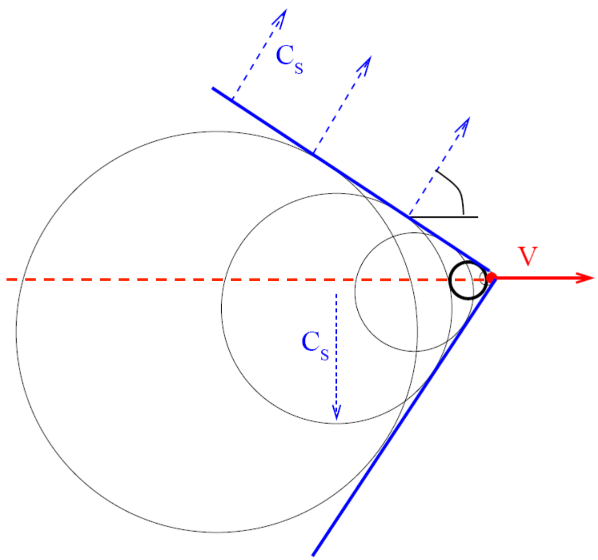
stronger wake

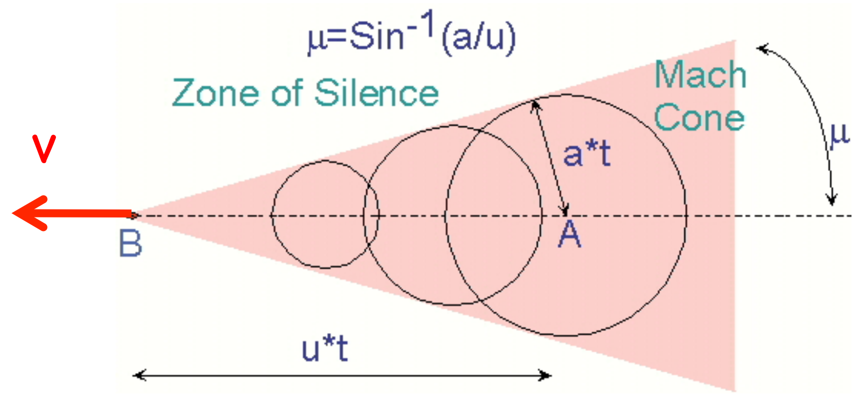


arXiv:0901.0230 nucl-th

I- Conical waves from supersonic motion

Spherical waves pile up and form a conical front





Introduce "stretched" coordinates

$$\begin{cases} \xi = \sigma^{1/2}\dfrac{(x - c_s t)}{R} \\[2ex] \tau = \sigma^{3/2}\dfrac{c_s t}{R} \end{cases}$$

Rewrite Euler and continuity eqs. as series of powers of σ

$$\sigma\Big(\ldots\Big) + \sigma^2\Big(\ldots\Big) + \sigma^3\Big(\ldots\Big) + \ldots = 0$$

each $\Big(\ldots\Big)$ vanishes

Continuity equation:

$$\sigma\left\{\frac{\partial v_1}{\partial \xi} - \frac{\partial \rho_1}{\partial \xi}\right\} + \sigma^2\left\{\frac{\partial v_2}{\partial \xi} - \frac{\partial \rho_2}{\partial \xi} + \frac{\partial \rho_1}{\partial \tau} + \rho_1\frac{\partial v_1}{\partial \xi} + v_1\frac{\partial \rho_1}{\partial \xi} - {c_s}^2 v_1\frac{\partial v_1}{\partial \xi}\right\} = 0$$

Obtain non-linear wave equations and go back to original coordinates

Define dimensionless variables:

$$\hat{\rho} = \frac{\rho}{\rho_0} \qquad \hat{\varepsilon} = \frac{\varepsilon}{\varepsilon_0} \qquad \hat{v} = \frac{v}{c_S}$$

Expand Euler and Continuity equations around equilibrium:

$$\left\{\begin{array}{l} \hat{\rho} = 1 + \sigma\rho_1 + \sigma^2\rho_2 + \dots \\ \hat{\varepsilon} = 1 + \sigma\varepsilon_1 + \sigma^2\varepsilon_2 + \dots \\ \hat{v} = \sigma v_1 + \sigma^2 v_2 + \dots \end{array}\right.$$

$$\sigma < 1$$

$$\hat{\varepsilon} = 1 + \hat{\varepsilon}_1 + \dots$$

$$\varepsilon_0\hat{\varepsilon}_1 = \delta\varepsilon(x, y, z, t)$$

$$n_{\vec{k}} \equiv n_{\vec{k}}(T) = \frac{1}{1 + e^{(k - \frac{1}{3}\mu)/T}} \qquad \bar{n}_{\vec{k}} \equiv \bar{n}_{\vec{k}}(T) = \frac{1}{1 + e^{(k + \frac{1}{3}\mu)/T}}$$

$$\gamma_G = 2(\text{polarizations}) \times 8(\text{colors}) = 16$$

$$\gamma_Q = 2(\text{spins}) \times 2(\text{flavors}) \times 3(\text{colors}) = 12$$

3-dimensional cylindrical case: (r, φ, z, t)

$$\hat{v}_r = \frac{v_r}{c_s}$$

$$\hat{\varepsilon} = \frac{\varepsilon}{\varepsilon_0}$$

$$\hat{v} = \frac{v}{c_s} = \frac{\sqrt{{v_r}^2 + {v_\varphi}^2 + {v_z}^2}}{c_s}$$

$$\hat{v}_z = \frac{v_z}{c_s} \qquad \hat{v}_\varphi = \frac{v_\varphi}{c_s}$$

Stretched coordinates:

$$R = \frac{\sigma^{1/2}}{L}(r - c_s t) \ , \quad \Phi = \sigma^{-1/2}\varphi \ , \quad Z = \frac{\sigma}{L} z \ , \quad T = \frac{\sigma^{3/2}}{L} c_s t$$

$$\hat{\varepsilon} = 1 + \sigma\varepsilon_1 + \sigma^2\varepsilon_2 + \sigma^3\varepsilon_3 + \ldots$$

and expansions:

$$\hat{v_r} = \sigma v_{r_1} + \sigma^2 v_{r_2} + \sigma^3 v_{r_3} + \ldots$$

$$\hat{v_\varphi} = \sigma^{3/2} v_{\varphi_1} + \sigma^{5/2} v_{\varphi_2} + \sigma^{7/2} v_{\varphi_3} + \ldots$$

$$\hat{v_z} = \sigma^{3/2} v_{z_1} + \sigma^{5/2} v_{z_2} + \sigma^{7/2} v_{z_3} + \ldots$$

The general nonlinear wave equation:

$$\frac{\partial}{\partial r}\left\{\frac{\partial \hat{\varepsilon}_1}{\partial t} + c_s \frac{\partial \hat{\varepsilon}_1}{\partial r} + \alpha \hat{\varepsilon}_1 \frac{\partial \hat{\varepsilon}_1}{\partial r} - \nu \frac{\partial^2 \hat{\varepsilon}_1}{\partial r^2} + \frac{\hat{\varepsilon}_1}{2t}\right\} + \frac{1}{2c_s t^2}\frac{\partial^2 \hat{\varepsilon}_1}{\partial \varphi^2} + \frac{c_s}{2}\frac{\partial^2 \hat{\varepsilon}_1}{\partial z^2} = 0$$

Where:

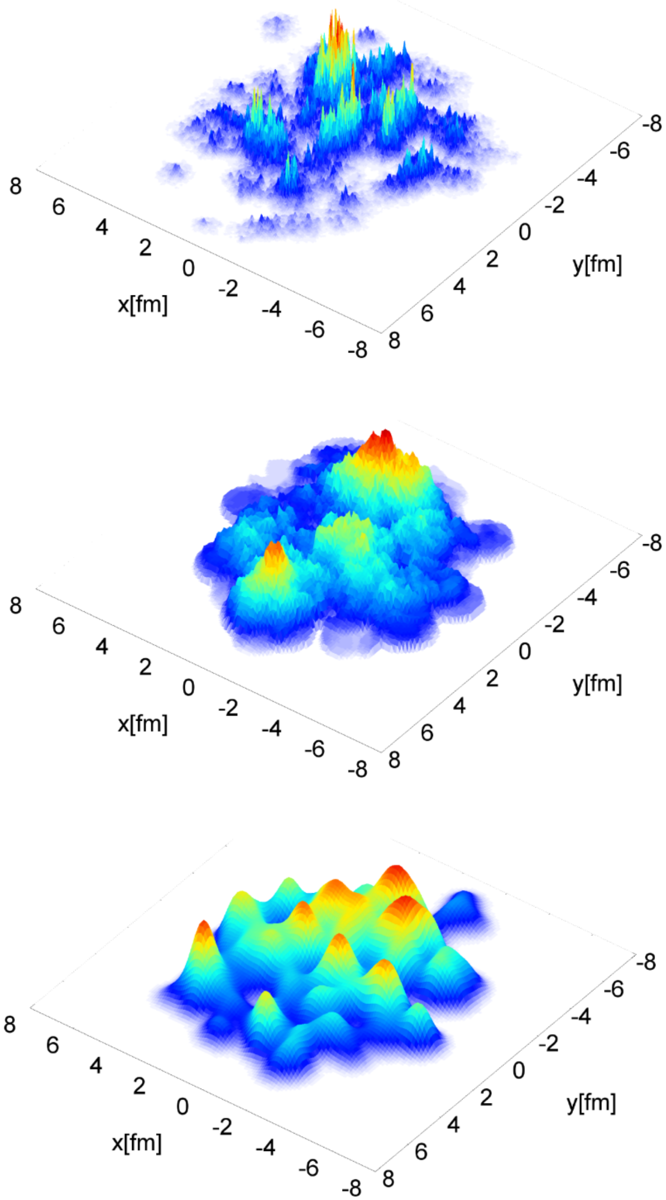
$$\alpha \equiv \frac{c_s}{2}\left[1 + \left(\frac{T_B}{T_0}\right)^4\right] \qquad \nu \equiv \frac{1}{2T_0}\left(\frac{\zeta}{s} + \frac{4}{3}\frac{\eta}{s}\right)$$

has four analytical solutions:

$$\boxed{\begin{aligned}\hat{\varepsilon}_1(r,z,\varphi,t) = {} & \frac{2\delta A}{c_s T_0}\left(\frac{\zeta}{s} + \frac{4}{3}\frac{\eta}{s}\right)\left[1 + \left(\frac{T_B}{T_0}\right)^4\right]^{-1} \\ & - \frac{2\delta A}{c_s T_0}\left(\frac{\zeta}{s} + \frac{4}{3}\frac{\eta}{s}\right)\left[1 + \left(\frac{T_B}{T_0}\right)^4\right]^{-1} \times \\ & \times tanh\left\{\delta\left[Ar + Bz - A\frac{c_s \varphi^2 t}{2} - \left(Ac_s + \frac{B^2 c_s}{2A} + \frac{\delta A^2}{T_0}\left(\frac{\zeta}{s} + \frac{4}{3}\frac{\eta}{s}\right)\right)t\right]\right\}\end{aligned}}$$

$$\boxed{\begin{aligned}\hat{\varepsilon}_1(r,z,\varphi,t) = -\frac{2\delta A}{c_s T_0}\left(\frac{\zeta}{s}+\frac{4}{3}\frac{\eta}{s}\right)\left[1+\left(\frac{T_B}{T_0}\right)^4\right]^{-1} \\ -\frac{2\delta A}{c_s T_0}\left(\frac{\zeta}{s}+\frac{4}{3}\frac{\eta}{s}\right)\left[1+\left(\frac{T_B}{T_0}\right)^4\right]^{-1}\times \\ \times\, tanh\left\{\delta\left[Ar+Bz-A\frac{c_s\varphi^2 t}{2}-\left(Ac_s+\frac{B^2 c_s}{2A}-\frac{\delta A^2}{T_0}\left(\frac{\zeta}{s}+\frac{4}{3}\frac{\eta}{s}\right)\right)t\right]\right\}\end{aligned}}$$

$$\boxed{\begin{aligned}\hat{\varepsilon}_1(r,z,\varphi,t) = \frac{2\delta A}{c_s T_0}\left(\frac{\zeta}{s}+\frac{4}{3}\frac{\eta}{s}\right)\left[1+\left(\frac{T_B}{T_0}\right)^4\right]^{-1} \\ -\frac{2\delta A}{c_s T_0}\left(\frac{\zeta}{s}+\frac{4}{3}\frac{\eta}{s}\right)\left[1+\left(\frac{T_B}{T_0}\right)^4\right]^{-1}\times \\ \times\, coth\left\{\delta\left[Ar+Bz-A\frac{c_s\varphi^2 t}{2}-\left(Ac_s+\frac{B^2 c_s}{2A}+\frac{\delta A^2}{T_0}\left(\frac{\zeta}{s}+\frac{4}{3}\frac{\eta}{s}\right)\right)t\right]\right\}\end{aligned}}$$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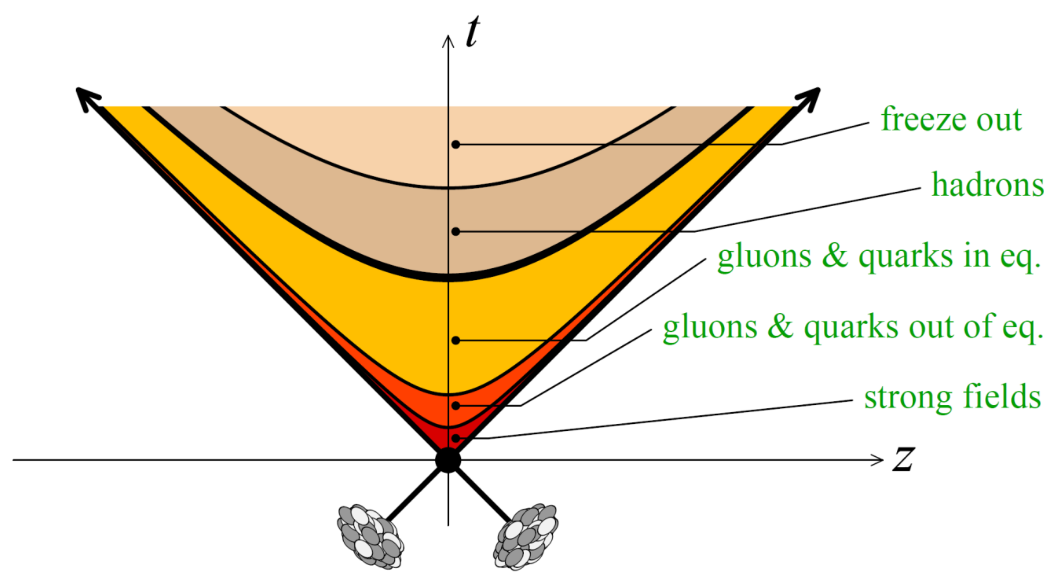
t
z
freeze out
hadrons
gluons & quarks in eq.
gluons & quarks out of eq.
strong fields