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Spin effects in relativistic hydrodynamics

OUTLINE

- Introduction
- Global thermodynamical equilibrium: rotating fluids
- Local thermodynamical equilibrium: thermo-vorticous fluids
- Polarization of spin ½ fermions in a thermo-vorticous flow
- The role of the spin tensor
- Outlook

Relativistic Aspects of Nuclear Physics
Rio de Janeiro, Sept. 23 2013

Cento di questi giorni!

Abstract

Investigation of the relevance of spin in RH

F.B., F. Piccinini, Ann. Phys. 323 (2008) 2452
F.B., L. Tinti, Ann. Phys. 325 (2010) 1566
F. B., L. Tinti, Phys. Rev. D 84 (2011) 025013
F. B., L. Tinti, Phys. Rev. D 87 (2013) 025029
F. B., V. Chandra, L. Del Zanna, E. Grossi, Ann. Phys. 338 (2013) 32 (*)
F.B., L. Csernai, D.J. Wang, arXiv:1304.4427, to appear in PRC

MAIN RESULT (*)

$$n_F = \frac{1}{e^{\beta(x)\cdot p - \xi(x)} + 1}$$

$$\Pi_\mu(p) = \epsilon_{\mu\rho\sigma\tau} \, \frac{p^\tau}{8m} \frac{\int d\Sigma_\lambda p^\lambda \, n_F (1 - n_F) \partial^\rho \beta^\sigma}{\int d\Sigma_\lambda p^\lambda \, n_F}$$

Polarization vector of a spin ½ particle in a relativistic fluid with inverse four-temperature $\beta = (1/T)\, u$ field at a freeze-out hypersurface Σ

Introduction

The single particle distribution function at local thermodynamical equilibrium (known as Juttner distribution) reads (spinless bosons):

$$f(x,p) = \frac{1}{e^{\beta(x)\cdot p - \xi(x)} - 1}$$

$$\beta^\mu = \frac{1}{T_0} u^\mu \qquad \xi = \mu_0 / T_0$$

In HIC often used, e.g., in the so-called Cooper-Frye formula:

$$\varepsilon \frac{\mathrm{d}N}{\mathrm{d}^3\mathrm{p}} = \int_\Sigma \mathrm{d}\Sigma_\mu p^\mu f(x,p)$$

QUESTION: *What happens if particles have a spin?*

Is it only a (2S+1) factor?
Answering this question urges us to review several of the "familiar" concepts of statistical mechanics and hydrodynamics. Quantum features cannot be neglected.

What is the distribution function?

Cannot say “the density of particles in phase space” because it does not take into account polarization degrees of freedom.
The answer can be found in the book: S.R. De Groot et al. *Relativistic kinetic theory*

Covariant Wigner function: scalar field

$$\langle \ \rangle = \mathrm{tr}(\widehat{\rho} \ )$$

$$W(x,k) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 y \, 2 \, \mathrm{e}^{-ik\cdot y} \langle : \widehat{\psi}^\dagger (x + y/2) \, \widehat{\psi}(x - y/2) : \rangle$$

For quasi-free theory, neglecting Compton-wavelength scale variations

$$W^+(x,k) \equiv \theta(k^0) W(x,k) = \int \frac{\mathrm{d}^3 \mathrm{p}}{\varepsilon} \, \delta^4(k-p) f(x,p)$$

$$W^-(x,k) \equiv \theta(-k^0) W(x,k) = \int \frac{\mathrm{d}^3 \mathrm{p}}{\varepsilon} \, \delta^4(k+p) \bar{f}(x,p)$$

which *define* the distribution functions of particles and antiparticles

Wigner function of the free Dirac field

$$W(x,k)_{AB} = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4y\, \mathrm{e}^{-ik\cdot y}\langle : \Psi_A(x-y/2)\overline{\Psi}_B(x+y/2) :\rangle$$
$$= \frac{1}{(2\pi)^4}\int \mathrm{d}^4y\, \mathrm{e}^{-ik\cdot y}\langle : \overline{\Psi}_B(x+y/2)\Psi_A(x-y/2) :\rangle$$

$$W^+(x,k) \equiv \theta(k^0)W(x,k) = \frac{1}{2}\int \frac{\mathrm{d}^3\mathrm{p}}{\varepsilon}\,\delta^4(k-p)\sum_{r,s} u_r(p) f_{rs}(x,p)\bar{u}_s(p)$$

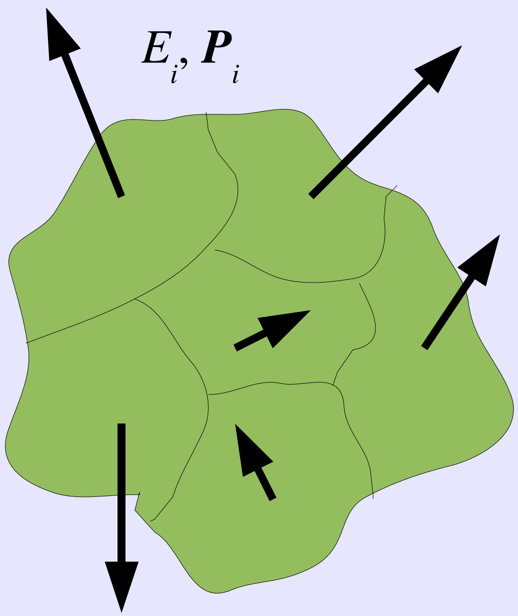
$$W^-(x,k) \equiv \theta(-k^0)W(x,k) = -\frac{1}{2}\int \frac{\mathrm{d}^3\mathrm{p}}{\varepsilon}\,\delta^4(k+p)\sum_{r,s} v_s(p) \bar{f}_{rs}(x,p)\bar{v}_r(p)$$

The *u, v* spinors are the usual solution of the free Dirac equation, with all of their well known properties (orthogonality and completeness).

Thus, the distribution function for spin ½ particles is a 2x2 matrix

Global thermodynamical equilibrium with angular momentum

non-quantum Landau's argument

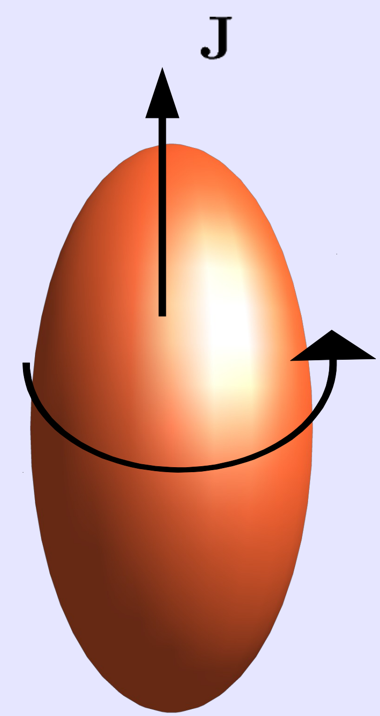


$$S = \sum_i S_i\left(\sqrt{E_i^2 - \mathbf{P}_i^2}\right) \qquad \frac{\partial S_i}{\partial E_i} = \frac{E_i}{M_i}\frac{\partial S_i}{\partial M_i} = \frac{\gamma_i}{T_i}$$

Maximize entropy with constraints

$$\sum_i S_i - \frac{\boldsymbol{\beta}}{T}\cdot\sum_i \mathbf{P}_i - \frac{1}{T}\left(\sum_i E_i - E_0\right) - \frac{\boldsymbol{\omega}}{T}\cdot\left(\sum_i \mathbf{x}_i \times \mathbf{P}_i - \mathbf{J}\right)$$

$$\frac{\gamma_i}{T_i} = \frac{1}{T} \quad \forall i \qquad \mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{x}_i \quad \forall i$$



Local temperature

$$T_i = \frac{T}{\sqrt{1 - (\boldsymbol{\omega} \times \mathbf{x}_i)^2}}$$

The quantum case

Density operator (see e.g. Landau, *Statistical physics*; A. Vilenkin, Phys. Rev. D 21 2260)

$$\widehat{\rho} = \frac{1}{Z_\omega} \exp\left[-\widehat{H}/T + \boldsymbol{\omega} \cdot \widehat{\mathbf{J}}/T + \mu \widehat{Q}/T \right]$$

Grand-canonical rotational partition function

Obtained by maximizing the entropy $S = -\mathrm{tr}(\widehat{\rho} \log \widehat{\rho})$ with respect to $\widehat{\rho}$
with the constraints of total mean energy, mean momentum and mean angular momentum
Fixed (equivalent to exact conservation for a *large* system)

ω/T is the Lagrange multiplier of the angular momentum conservation constraint
and its physical meaning is that of an *angular velocity*

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$$

Single particle distribution function at global thermodynamical equilibrium

In the Boltzmann limit, for an ideal relativistic gas, this is a calculation which can be done without the explicit use of quantum field theory, just with quantum statistical mechanics and group theory (F. B., L. Tinti, Ann. Phys. 325, 1566 (2010)).

More explicitely: maximal entropy (equipartition), angular momentum conservation and Lorentz group representation theory.

$$f(x,p)_{rs} = \mathrm{e}^{\xi}\, \mathrm{e}^{-\beta\cdot p}\ \frac{1}{2}\left(D^S([p]^{-1}\mathsf{R}_{\hat{\omega}}(i\omega/T)[p]) + D^S([p]^{\dagger}\mathsf{R}_{\hat{\omega}}(i\omega/T)[p]^{\dagger -1})\right)_{rs}$$

$\mathsf{R}_{\hat{\omega}}(i\omega/T) = \exp[D^S(J_3)\omega/T]$ = SL(2,C) matrix representing a rotation around $\boldsymbol{\omega}$ axis (z or 3) by an imaginary angle iω/T.

$$\mathrm{tr}_{2S+1} f = \mathrm{e}^{\xi}\, \mathrm{e}^{-\beta\cdot p}\mathrm{tr}_{2S+1}\mathsf{R}_{\hat{\omega}}(i\omega/T) = \mathrm{e}^{\xi}\, \mathrm{e}^{-\beta\cdot p}\sum_{\sigma=-S}^{S} \mathrm{e}^{-\sigma\omega/T} \equiv \mathrm{e}^{\xi}\, \mathrm{e}^{-\beta\cdot p}\chi\left(\frac{\omega}{T}\right)$$

As a consequence, particles with spin get polarized in a rotating gas

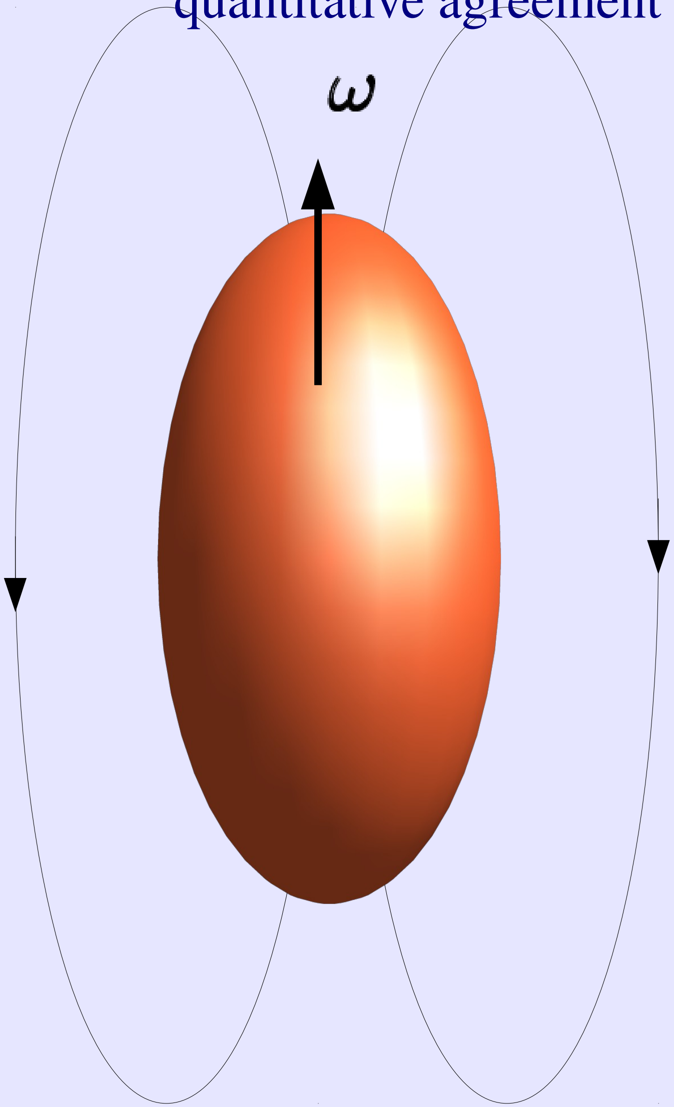
$$\mathbf{\Pi}_0 = \frac{\sum_{n=-S}^{S} n \mathrm{e}^{n\omega/T}}{\sum_{n=-S}^{S} \mathrm{e}^{n\omega/T}}\left[\frac{\varepsilon}{m}\hat{\boldsymbol{\omega}} - \frac{\hat{\boldsymbol{\omega}}\cdot\mathbf{p}\mathbf{p}}{m(\varepsilon+m)}\right]$$

F.B., F. Piccinini, Ann. Phys. 323, 2452 (2008)

Barnett effect

S. J. Barnett, Magnetization by Rotation, Phys. Rev.. 6, 239–270 (1915).

Spontaneous magnetization of an uncharged body when spun around its axis, in quantitative agreement with the previous polarization formula



$$M = \frac{\chi}{g}\omega$$

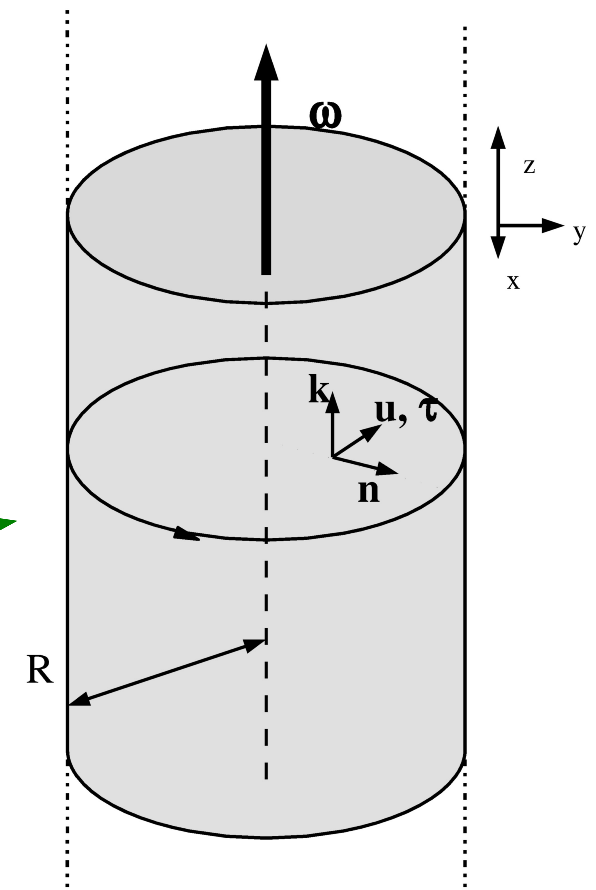
It is a dissipative transformation of the orbital angular momentum into spin of the constituents. The angular velocity decreases and a small magnetic field appears; this phenomenon is accompanied by a heating of the sample. Requires a spin-orbit coupling.

Dirac-ization of f

For the case S=1/2 the formulae can be rewritten using Dirac spinors

$$f(x,p) = \mathrm{e}^{\xi}\,\mathrm{e}^{-\beta\cdot p}\;\frac{1}{2m}\bar{U}(p)\exp[(\omega/T)\,\Sigma_z]U(p) \qquad \Sigma_z = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

$$\bar{f}(x,p) = -\mathrm{e}^{-\xi}\,\mathrm{e}^{-\beta\cdot p}\;\frac{1}{2m}\left[\bar{V}(p)\exp[-(\omega/T)\,\Sigma_z]V(p)\right]^T$$



They can be also rewritten in a fully covariant form taking into account that

$$\varpi_{\mu\nu} = (\omega/T)(\delta^1_\mu\delta^2_\nu - \delta^1_\nu\delta^2_\mu) = \sqrt{\beta^2}\,\Omega_{\mu\nu}$$

Ω being the acceleration tensor of the Frenet-Serret tetrad of the velocity field lines

and the generators of the Lorentz group representation

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]$$

$$f(x,p) = \mathrm{e}^{\xi}\,\mathrm{e}^{-\beta\cdot p}\;\frac{1}{2m}\bar{U}(p)\exp\left[\frac{1}{2}\varpi^{\mu\nu}\Sigma_{\mu\nu}\right]U(p) \qquad \bar{f}(x,p) = -\mathrm{e}^{-\xi}\,\mathrm{e}^{-\beta\cdot p}\;\frac{1}{2m}\left[\bar{V}(p)\exp\left[-\frac{1}{2}\varpi^{\mu\nu}\Sigma_{\mu\nu}\right]V(p)\right]^T$$

Single particle distribution function at local thermodynamical equilibrium

In principle, it should be calculated from the covariant Wigner function with the local thermodynamical equilibrium quantum density operator

$$\widehat{\rho}_{\rm LE}(t) = \frac{\exp[-\int \mathrm{d}^3\mathrm{x} \, \left(\widehat{T}^{0\nu}\beta_\nu(x) - \widehat{j}^0\xi(x) - \frac{1}{2}\widehat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(x)\right)]}{\mathrm{tr}(\exp[-\int \mathrm{d}^3\mathrm{x} \, \left(\widehat{T}^{0\nu}\beta_\nu(x) - \widehat{j}^0\xi(x) - \frac{1}{2}\widehat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(x)\right)])}$$

Obtained by maximizing the entropy $S = -\mathrm{tr}(\widehat{\rho}\log\widehat{\rho})$ with respect to $\widehat{\rho}$ with the constraints of fixed mean energy-momentum density and fixed mean angular momentum density.

$$W(x,k) = \mathrm{tr}(\widehat{\rho}_{LE}(t)\text{Combination of quantum fields})$$

A complicated calculation...

One can make a reasonable ansatz which

- reduces to the global equilibrium solution in the Boltzmann limit
- reduces to the known Fermi-Juttner or Bose-Juttner formulae at the LTE in the non-rotating case

Ansatz for LTE

$$f(x,p) = \frac{1}{2m}\bar{U}(p)\left(\exp[\beta(x)\cdot p - \xi(x)]\exp[-\frac{1}{2}\varpi(x):\Sigma] + I\right)^{-1} U(p)$$

$$\bar{f}(x,p) = -\frac{1}{2m}(\bar{V}(p)\left(\exp[\beta(x)\cdot p + \xi(x)]\exp[\frac{1}{2}\varpi(x):\Sigma] + I\right)^{-1} V(p))^T$$

Example:

Recalling: $U(p)\bar{U}(p) = (\not{p} + m)I \qquad V(p)\bar{V}(p) = (\not{p} - m)I$

$$\mathrm{tr}_2 f = \frac{1}{2m}\mathrm{tr}_2(\bar{U}(p)XU(p)) = \frac{1}{2m}\mathrm{tr}(XU(p)\bar{U}(p)) = \frac{1}{2m}\mathrm{tr}(X(\not{p}+m)) = \frac{1}{2}\mathrm{tr}X$$

$$\mathrm{tr}_2 \bar{f} = -\frac{1}{2m}\mathrm{tr}_2(\bar{V}(p)\bar{X}V(p)) = -\frac{1}{2m}\mathrm{tr}(\bar{X}V(p)\bar{V}(p)) = -\frac{1}{2m}\mathrm{tr}(\bar{X}(\not{p}-m)) = \frac{1}{2}\mathrm{tr}\bar{X}$$

with $$X = \left(\exp[\beta(x)\cdot p - \xi(x)]\exp\left[-\frac{1}{2}\varpi(x):\Sigma\right] + I\right)^{-1}$$

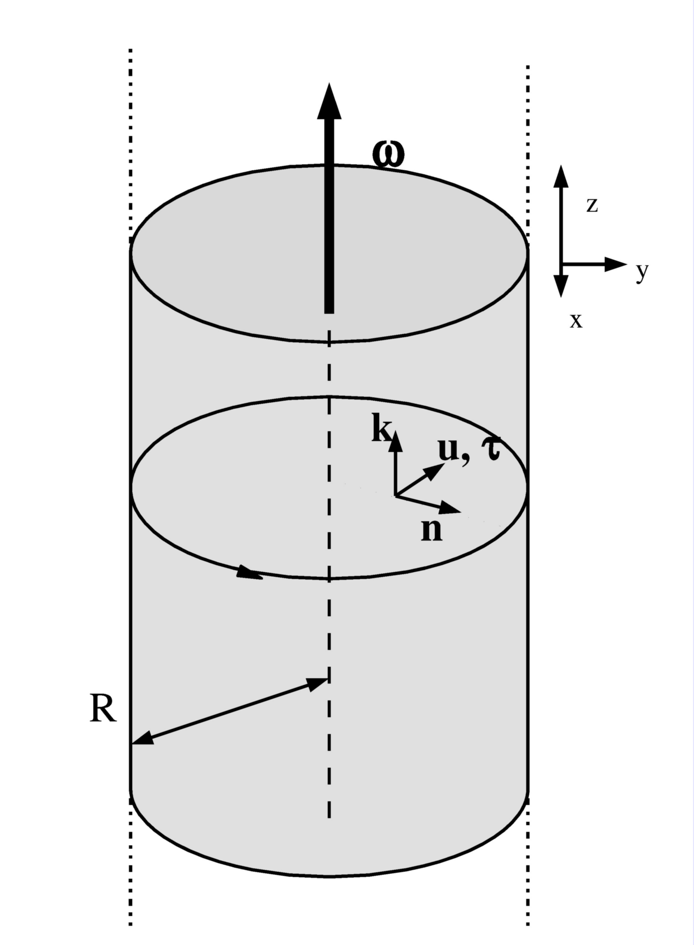
What is $\varpi(x)$?

This is a crucial issue to calculate polarization

At global equilibrium:

$$\varpi_{\mu\nu} = (\omega/T)(\delta^1_\mu \delta^2_\nu - \delta^1_\nu \delta^2_\mu) = \sqrt{\beta^2}\,\Omega_{\mu\nu}$$

$$\sqrt{\beta^2} = \frac{1}{T_0} \qquad \Omega^{\mu\nu} = \sum_{i=1}^{4} \frac{De^\mu_i}{\mathrm{d}\tau} e^{i\nu}$$



$$e_0 = u \quad e_1 = (0, \hat{\mathbf{n}}) \quad e_2 = (0, \hat{\mathbf{k}}) \quad e_3 = \tau$$

At the same time

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$$

being, for global equilibrium

$$\beta = \frac{1}{T}(1, \boldsymbol{\omega} \times \mathbf{x}) = \frac{1}{T_0}(\gamma, \gamma \mathbf{v})$$

The latter equation can be checked explicitely, but it is form is indeed a deeper consequence of relativity coupled with thermodynamics

Equilibrium in relativity can be achieved only if the inverse four-temperature field is a Killing vector

$$\boxed{\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0}$$

$$\Rightarrow \quad \beta_\mu = b_\mu + \varpi_{\mu\nu} x^\nu \qquad b \text{ and } \varpi \text{ constants}$$

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$$

$\Rightarrow$ ***"Thermal vorticity"***

If deviations from equilibrium are *small*, we know that the tensor $\varpi(x)$ should differ from the above expression only by terms which vanish at equilibrium, i.e. second-order terms in the gradients of the β field

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu) + \mathcal{O}(\partial^2 \beta)$$

This is what we need for leading-order hydrodynamics!

Polarization in a relativistic fluid

Definition:

$$\Pi_\mu = -\frac{1}{2}\epsilon_{\mu\rho\sigma\tau} S^{\rho\sigma} \frac{p^\tau}{m}$$

also known as Pauli-Lubanski vector

should be the total angular momentum vector of the particle

For a kinetic system

$$\langle \Pi_\mu(x,p) \rangle = -\frac{1}{2}\frac{1}{\mathrm{tr}_2 f}\epsilon_{\mu\rho\sigma\tau} \frac{\mathrm{d}\mathcal{J}^{0,\rho\sigma}(x,p)}{\mathrm{d}^3 p}\frac{p^\tau}{m}$$

Total angular momentum tensor

$$\mathcal{J}^{\lambda,\rho\sigma}(x) = x^\rho T^{\lambda\sigma}(x) - x^\sigma T^{\lambda\tau}(x) + \mathcal{S}^{\lambda,\rho\sigma}(x)$$

$$\frac{\mathrm{d}\mathcal{J}^{0,\rho\sigma}(x)}{\mathrm{d}^3 p} = (x^\rho p^\sigma - x^\sigma p^\rho)\mathrm{tr}_2 f(x,p) + \frac{\mathrm{d}\mathcal{S}^{\lambda,\rho\sigma}(x)}{\mathrm{d}^3 p}$$

vanished by the Levi-Civita symbol

Canonical spin tensor

$$\mathcal{S}^{\lambda,\mu\nu}(x) \equiv \frac{1}{2}\langle : \overline{\Psi}(x)\{\gamma^\lambda, \Sigma^{\mu\nu}\}\Psi(x) : \rangle = \frac{1}{2}\int \frac{\mathrm{d}^3\mathrm{p}}{2\varepsilon}\mathrm{tr}_2\left(f(x,p)\bar{U}(p)\{\gamma^\lambda, \Sigma^{\mu\nu}\}U(p)\right) - \mathrm{tr}_2\left(\bar{f}^T(x,p)\bar{V}(p)\{\gamma^\lambda, \Sigma^{\mu\nu}\}V(p)\right)$$

...tracing the γ's, expanding in $\varpi(x)$ which is usually a small number (at global equilibrium $\hbar\omega/KT \ll 1$)...

$$\frac{\mathrm{d}\mathcal{S}^{\lambda,\rho\sigma}(x)}{\mathrm{d}^3 p} \simeq \frac{1}{2\varepsilon}\left(p^\lambda n_F(1-n_F)\varpi^{\rho\sigma} + \text{rotation of indices}\right)$$

$$n_F = \frac{1}{\mathrm{e}^{\beta(x)\cdot p - \xi(x)} + 1}$$

Polarization four-vector in the LAB frame

Final formulae:

$$\langle \Pi_\mu(x,p)\rangle \simeq \frac{1}{16}\epsilon_{\mu\rho\sigma\tau}\,(1-n_F)\,(\partial^\rho\beta^\sigma - \partial^\sigma\beta^\rho)\,\frac{p^\tau}{m} = \frac{1}{8}\epsilon_{\mu\rho\sigma\tau}\,(1-n_F)\partial^\rho\beta^\sigma\frac{p^\tau}{m}$$

$$\Pi = (\Pi^0, \boldsymbol{\Pi}) = \frac{1-n_F}{8m}((\nabla\times\boldsymbol{\beta})\cdot\mathbf{p}, \varepsilon(\nabla\times\boldsymbol{\beta}) - \frac{\partial\boldsymbol{\beta}}{\partial t}\times\mathbf{p} - \nabla\beta^0\times\mathbf{p})$$

F. B., V. Chandra, L. Del Zanna, E. Grossi, arXiv:1303.3431 Ann. Phys. 338 (2013) 32

As a by-product, a new effect is predicted: particles in a steady temperature gradient (here with $\boldsymbol{v} = 0$) should be transversely polarized:

$$\Pi = (\Pi^0, \boldsymbol{\Pi}) = (1-n_F)\frac{\hbar p}{8mKT^2}(0, \nabla T\times\hat{\mathbf{p}})$$

Polarization in relativistic heavy ion collisions

There have been several papers in the past years about this subject:

A. Ayala et al., Phys. Rev. C 65 024902 (2002)
Z. T. Liang, X. N. Wang, Phys. Rev. Lett. 94 102301 (2005) and others
B. Betz, M. Gyulassy and G. Torrieri, Phys. Rev. C 76 044901 (2007)
F. B., F. Piccinini and J. Rizzo, Phys. Rev. C 77 024906 (2008)

yet no definite formula connecting the polarization of hadrons to the hydrodynamical model.

Now we have it:

$$\Pi_\mu(p) = \epsilon_{\mu\rho\sigma\tau} \; \frac{p^\tau}{8m} \frac{\int \mathrm{d}\Sigma_\lambda p^\lambda \, n_F (1 - n_F) \partial^\rho \beta^\sigma}{\int \mathrm{d}\Sigma_\lambda p^\lambda \, n_F}$$

and we can use it to predict Λ polarization in peripheral heavy ion collisions

Distribution of protons in the Λ rest frame

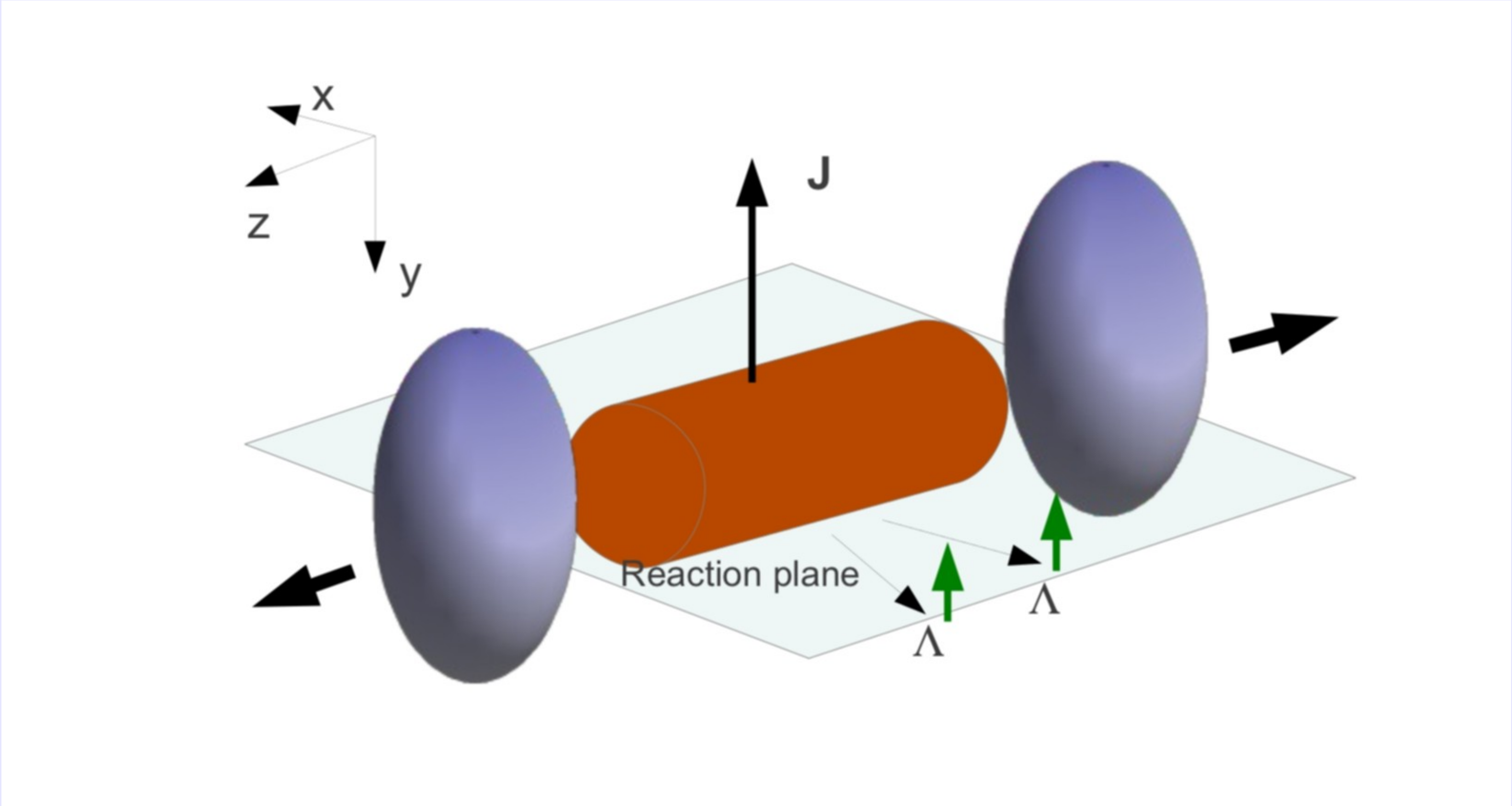
$$\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}\Omega^*} = \frac{1}{4\pi}\left(1 + \alpha\, \mathbf{\Pi}_0 \cdot \hat{\mathbf{p}}^*\right) \qquad \mathbf{\Pi}_0(p) = \mathbf{\Pi}(p) - \frac{\mathbf{p}}{\varepsilon(\varepsilon + m)} \mathbf{\Pi}(p) \cdot \mathbf{p}$$

Because of the parity symmetry of the collision

$$\mathbf{\Pi}(p) = \frac{\varepsilon}{8m} \frac{\int \mathrm{d}V \, n_F (\nabla \times \boldsymbol{\beta})}{\int \mathrm{d}V \, n_F}$$

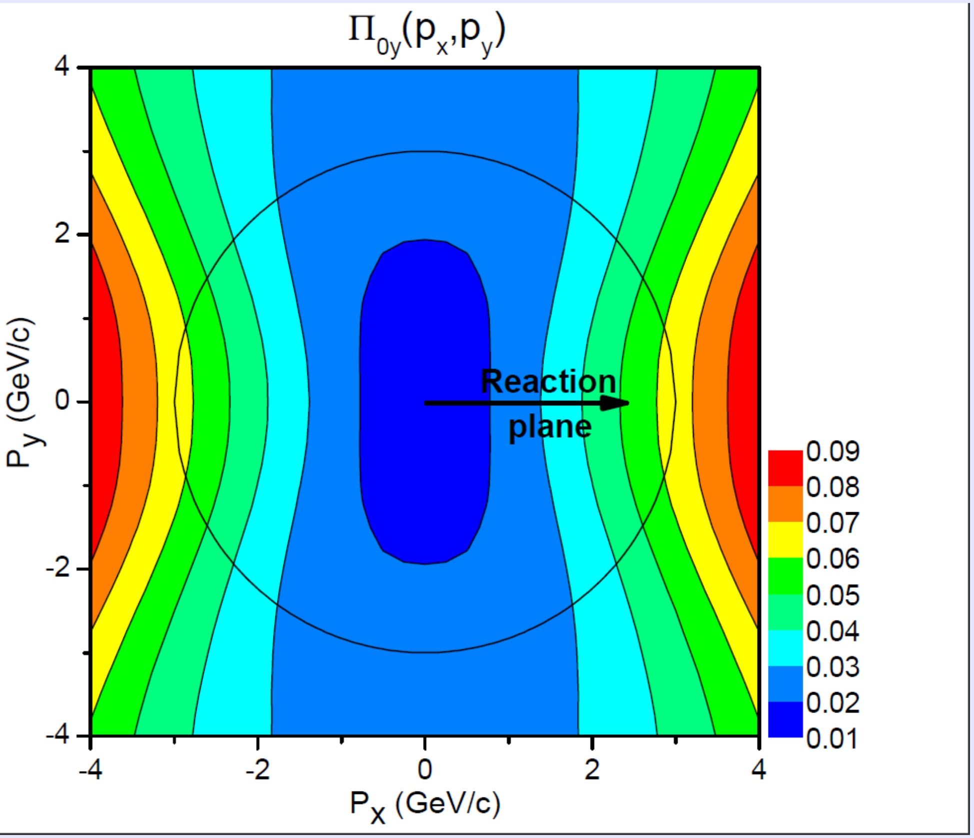
The most polarized Λ are those in the reaction plane (normal to angular momentum).

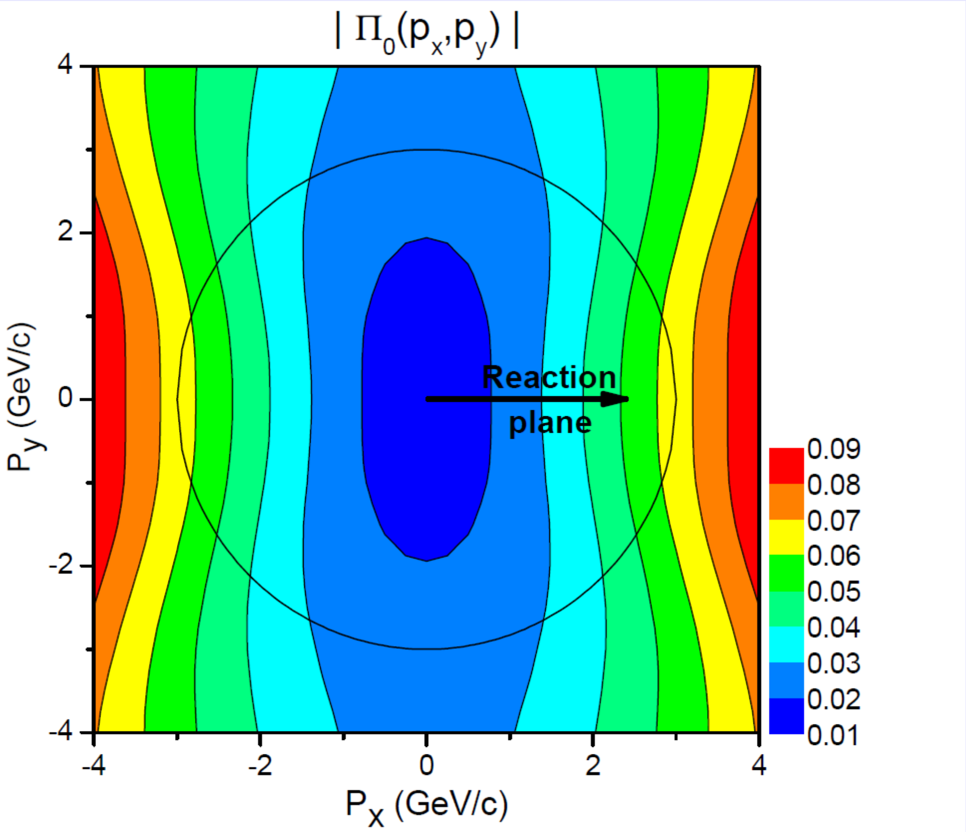
DISTINCTIVE FEATURE: particle and antiparticle polarization have the same orientation, unlike in a magnetic field



The amount of polarization depends on the thermo-vorticity field

Calculations based on: L. Csernai, V. Magas, D.J. Wang, Phys. Rev. C 87 034906 (2013)





F.B., L. Csernai, D.J. Wang arXiv:1304.4427, PRC in press

Average polarization consistent with the bound set by RHIC (<0.02).

NOTE: the polarization owing to the spectator's magnetic field (E. Bratkovskaya et al.) is at least 4 orders of magnitude less than the one shown above

The role of the spin tensor in relativistic hydrodynamics

F. B., L. Tinti, Phys. Rev. D 84 (2011) 025013
F. B., L. Tinti, Phys. Rev. D 87 (2013) 025029

In Minkowski space-time, from translational and Lorentz invariance one obtains two conserved Noether currents:

$$\widehat{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^a)} \partial^\nu \Psi^a - g^{\mu\nu} \mathcal{L}$$

$$\widehat{\mathcal{S}}^{\lambda,\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \Psi^a)} D^A (J^{\mu\nu})^a_b \Psi^b$$

It is very important to stress that these are <u>operators</u> (henceforth denoted with a hat)

$$\partial_\mu \widehat{T}^{\mu\nu} = 0$$

$$\partial_\lambda \widehat{\mathcal{J}}^{\lambda,\mu\nu} = \partial_\lambda \left(\widehat{\mathcal{S}}^{\lambda,\mu\nu} + x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu} \right) = \partial_\lambda \widehat{\mathcal{S}}^{\lambda,\mu\nu} + \widehat{T}^{\mu\nu} - \widehat{T}^{\nu\mu} = 0$$

Relation between macroscopic (classical) and quantum observables

$$\mathcal{O}_{\rm cl.} = {\rm tr}\left(\widehat{\rho} : \widehat{O} :\right)$$

Therefore we define as the (macroscopic) stress-energy and spin tensors:

$$T^{\mu\nu}(x) = {\rm tr}\left(\widehat{\rho} : \widehat{T}^{\mu\nu}(x) :\right)$$

$$\mathcal{S}^{\lambda,\mu\nu}(x) = {\rm tr}\left(\widehat{\rho} : \widehat{\mathcal{S}}^{\lambda,\mu\nu}(x) :\right)$$

Pseudo-gauge transformations with a *superpotential* $\widehat{\Phi}$

F.W. Hehl, Rep. Mat. Phys. 9 (1976) 55

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha \left(\widehat{\Phi}^{\alpha,\mu\nu} - \widehat{\Phi}^{\mu,\alpha\nu} - \widehat{\Phi}^{\nu,\alpha\mu} \right)$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = \widehat{\mathcal{S}}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu} + \partial_\alpha \widehat{\Xi}^{\alpha\lambda,\mu\nu}$$

With (we confine ourselves to $\Xi = 0$):

$$\int_{\partial\Omega} \mathrm{d}S \left(\widehat{\Phi}^{i,0\nu} - \widehat{\Phi}^{0,i\nu} - \widehat{\Phi}^{\nu,i0} \right) n_i = 0$$

$$\int_{\partial\Omega} \mathrm{d}S \left[x^\mu \left(\widehat{\Phi}^{i,0\nu} - \widehat{\Phi}^{0,i\nu} - \widehat{\Phi}^{\nu,i0} \right) - x^\nu \left(\widehat{\Phi}^{i,0\mu} - \widehat{\Phi}^{0,i\mu} - \widehat{\Phi}^{\mu,i0} \right) \right] n_i = 0$$

They leave the conservation equations and spacial integrals (=generators, or total energy, momentum and angular momentum) invariant.

This seems to be enough for a quantum relativistic field theory. It is not in gravity but, as long as we disregard it, different couples of tensors related by a pseudo-gauge transformation cannot be distinguished

Example: Belinfante symmetrization procedure

Just take $\widehat{\Phi} = \widehat{\mathcal{S}}$

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha \left(\widehat{\mathcal{S}}^{\alpha,\mu\nu} - \widehat{\mathcal{S}}^{\mu,\alpha\nu} - \widehat{\mathcal{S}}^{\nu,\alpha\mu} \right)$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = 0$$

This is a way of getting rid of the spin tensor, whose physical meaning seems to be thus very limited in QFT (eliminated by a pseudo-gauge transformation). The (mean value of the) above symmetrized Belinfante tensor is commonly assumed to be the source of the gravitational field, at least in GR.

Nevertheless, if we are interested in local mean values, that is mean energy-momentum density and angular momentum density or polarization, the equivalence may be broken. Thermodynamics and hydrodynamics make a difference!

The free Dirac field in a rotating cylinder

$$\widehat{T}^{\mu\nu} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi$$

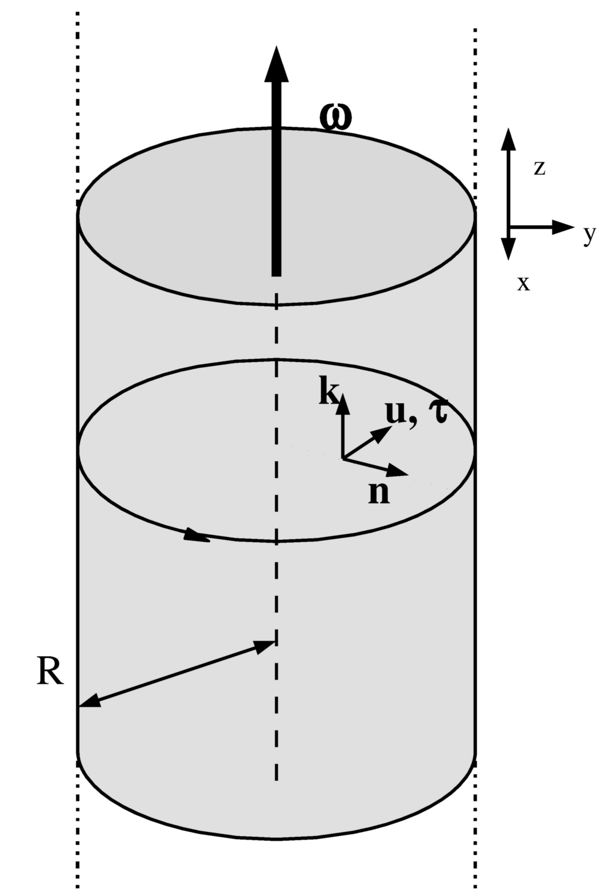
Canonical couple

$$\widehat{\mathcal{S}}^{\lambda,\mu\nu} = \frac{1}{2}\overline{\Psi}\{\gamma^\lambda, \Sigma^{\mu\nu}\}\Psi = \frac{i}{8}\overline{\Psi}\{\gamma^\lambda[\gamma^\mu,\gamma^\nu]\}\Psi$$

$$\widehat{T}'^{\mu\nu} = \frac{i}{4}\left[\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi + \overline{\Psi}\gamma^\nu \overleftrightarrow{\partial}^\mu \Psi\right]$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = 0$$

Belinfante couple



Belinfante set is obtained from the canonical with the superpotential $\widehat{\Phi} = \widehat{\mathcal{S}}_{\text{canonical}}$

NOTE: the canonical Dirac spin tensor is also antisymmetric in the first two indices

$$\widehat{\mathcal{S}}^{\lambda,\mu\nu} = -\widehat{\mathcal{S}}^{\mu,\lambda\nu}$$

$$\mathcal{S}^{\lambda,\mu\nu} = D(r)[(n^\mu\tau^\nu - n^\nu\tau^\mu)u^\lambda + (n^\lambda\tau^\mu - n^\mu\tau^\lambda)u^\nu - (n^\lambda\tau^\nu - n^\nu\tau^\lambda)u^\mu]$$

From previous expressions it follows

$$\mathcal{S}^{0,ij} = D(r)\epsilon_{ijk}\hat{k}^k$$

$$\begin{aligned} T^{0i}_{\rm Belinfante} &= T^{0i}_{\rm canonical} - \frac{1}{2}\partial_\alpha \mathcal{S}^{0,\alpha i} = T^{0i}_{\rm canonical} - \frac{1}{2}\partial_\alpha \epsilon_{\alpha i k} D(r)\hat{k}^k \\ &= T^{0i}_{\rm canonical} + \frac{1}{2}({\rm rot}\mathbf{D})^i = T^{0i}_{\rm canonical} - \frac{1}{2}\frac{{\rm d}D(r)}{{\rm d}r}\hat{v}^i \end{aligned}$$

$$\boldsymbol{\mathcal{J}}_{\rm Belinfante} = \boldsymbol{\mathcal{J}}_{\rm canonical} - \left(\frac{1}{2} r \frac{{\rm d}D(r)}{{\rm d}r} + D(r)\right)\hat{\mathbf{k}}$$

The momentum density and/or angular momentum density might differ in the canonical or Belinfante case if D(r) is non-vanishing

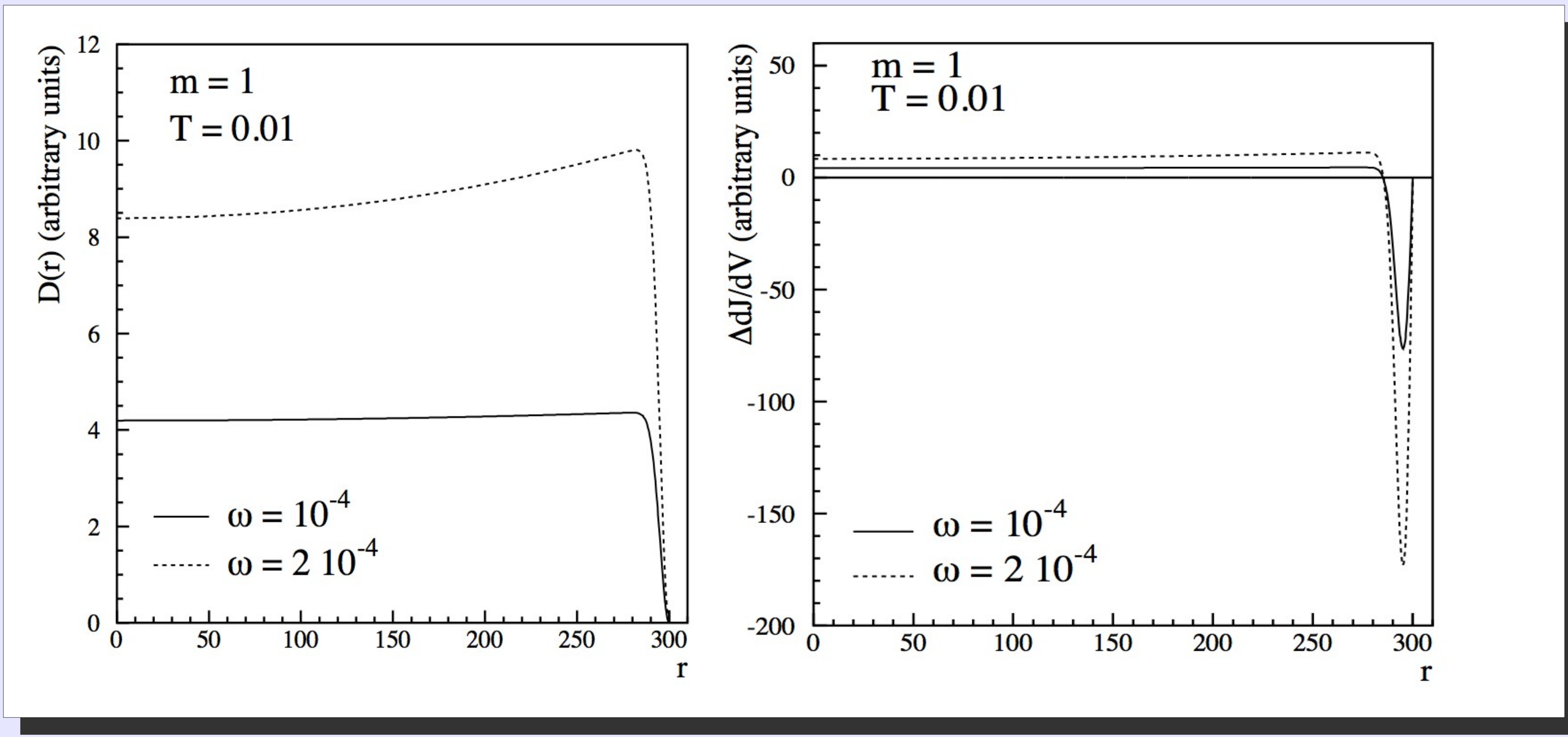
It can be shown that D(r) and its derivative are non vanishing, so the conclusion is the canonical and Belinfante set are thermodynamically inequivalent.

D(r) in the non-relativistic limit

It is the sum of a particle and antiparticle term:

$$D(r)^{\pm} = \hbar \mathrm{tr}\left[\widehat{\rho}\left(:\Psi^{\dagger}\Sigma_z\Psi:\right)^{\pm}\right] \simeq \frac{1}{2}\frac{\hbar\omega}{KT}\hbar \mathrm{tr}\left[\widehat{\rho}\left(:\Psi^{\dagger}\Psi:\right)^{\pm}\right] = \hbar\frac{1}{2}\frac{\hbar\omega}{KT}\left(\frac{\mathrm{d}n}{\mathrm{d}^3\mathbf{x}}\right)^{\pm}$$

we can make a numerical computation of the D(r) function:



Nonequilibrium inequivalence: Change of transport coefficients

Kubo formula for shear viscosity:

$$\eta = \lim_{\varepsilon\to 0}\lim_{\mathbf{k}\to 0} \,\mathrm{Im}\int_{-\infty}^{0} \mathrm{d}t\, \frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon} \int \mathrm{d}^3\mathrm{x}\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \langle \left[\widehat{T}_S^{12}(x), \widehat{T}_S^{12}(0)\right]\rangle_0$$

A change of the stress-energy tensor:

$$\widehat{T}_S^{\prime\mu\nu} = \widehat{T}_S^{\mu\nu} - \frac{1}{2}\partial_\lambda\left(\widehat{\Phi}^{\mu,\lambda\nu} + \widehat{\Phi}^{\nu,\lambda\mu}\right) = \widehat{T}_S^{\mu\nu} - \partial_\lambda \widehat{\Xi}^{\lambda\mu\nu}$$

Reflects into a change of shear viscosity

$$\begin{aligned}\Delta\eta = \eta' - \eta = & -\lim_{k\to 0}\int_V \mathrm{d}^3\mathrm{x}\, \cos kx^1 \langle\left[\widehat{\Xi}^{012}(0,\mathrm{x}), \widehat{\Xi}^{012}(0,\mathbf{0})\right]\rangle_0 \\ & - 2\lim_{\varepsilon\to 0}\lim_{k\to 0}\,\mathrm{Im}\int_{-\infty}^{0}\mathrm{d}t\, \mathrm{e}^{\varepsilon t}\int \mathrm{d}^3\mathrm{x}\, \mathrm{e}^{ikx^1}\langle\left[\widehat{\Xi}^{012}(x), \widehat{T}_S^{12}(0,\mathbf{0})\right]\rangle_0\end{aligned}$$

Note that the change of shear viscosity is not compensated by a change of another transport coefficient so as to maintain the same entropy production rate: also entropy changes!

Conclusions and Outlook

- We have determined the relativistic distribution function of particles with spin ½ at local thermodynamical equilibrium.

- This formula allows to *quantitatively* determine polarization of baryons in peripheral relativistic heavy ion collisions at the freeze-out and its momentum dependence.

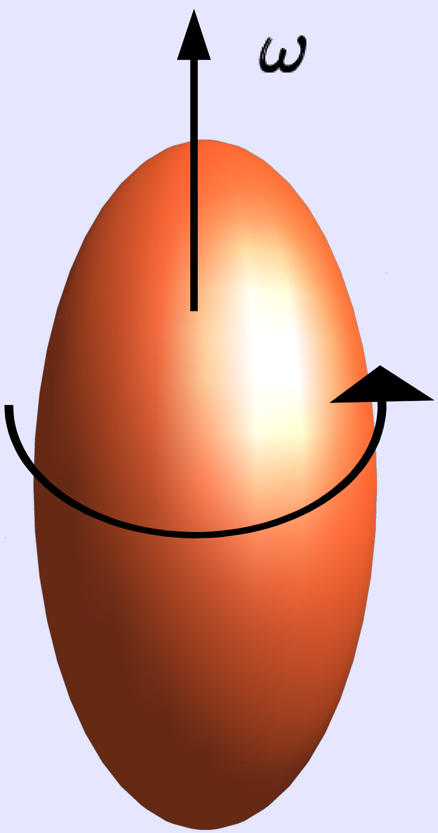
- The detection of a polarization (in agreement with the prediction of the hydro model) would be a striking confirmation of the local thermodynamical equilibrium picture and, to my knowledge, it would be the first direct observation of polarization induced by rotation for single particles (Barnett effect sees the induced B field)

The role of spin degrees of freedom in relativistic hydrodynamics is definitely worth being investigated. It may lead to the observation of intriguing phenomena and have even more intriguing connections with fundamental physics

Barnett effect

S. J. Barnett, Magnetization by Rotation, Phys. Rev.. 6, 239–270 (1915).

Spontaneous magnetization of an uncharged body when spun around its axis, in quantitative agreement with the previous polarization formula



$$M = \frac{\chi}{g}\omega$$

It is a dissipative transformation of the orbital angular momentum into spin of the constituents. The angular velocity decreases and a small magnetic field appears; this phenomenon is accompanied by a heating of the sample.

Converse: Einstein-De Haas effect

the only experiment by Einstein

A. Einstein, W. J. de Haas, Koninklijke Akademie van Wetenschappen te Amsterdam, Proceedings, 18 I, 696-711 (1915)

Rotation of a ferromagnet originally at rest when put into an external H field

An effect of angular momentum conservation:
spins get aligned with H (irreversibly) and this must be compensated by a on overall orbital angular momentum