The long-wavelength limit of the Boltzmann equation: recent insights in deriving dissipative relativistic fluid dynamics

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Fluid dynamics: degrees of freedom

1. **Net charge current:** $\boxed{N^\mu = n\, u^\mu + n^\mu}$

   u^μ **fluid 4-velocity,** $u^\mu u_\mu = u^\mu g_{\mu\nu} u^\nu = 1$

   $g_{\mu\nu} \equiv \mathrm{diag}(+,-,-,-)$ **(West coast!!) metric tensor**

   $n \equiv u^\mu N_\mu$ **net charge density in fluid rest frame**

   $n^\mu \equiv \Delta^{\mu\nu} N_\nu \equiv N^{<\mu>}$ **diffusion current (flow of net charge relative to u^μ),** $n^\mu u_\mu = 0$

   $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ **projector onto 3-space orthogonal to u^μ,** $\Delta^{\mu\nu} u_\nu = 0$

2. **Energy-momentum tensor:** $\boxed{T^{\mu\nu} = \epsilon\, u^\mu u^\nu - (p + \Pi)\, \Delta^{\mu\nu} + 2\, q^{(\mu} u^{\nu)} + \pi^{\mu\nu}}$

   $\epsilon \equiv u^\mu T_{\mu\nu} u^\nu$ **energy density in fluid rest frame**

   p **pressure in fluid rest frame**

   Π **bulk viscous pressure,** $p + \Pi \equiv -\frac{1}{3}\, \Delta^{\mu\nu} T_{\mu\nu}$

   $q^\mu \equiv \Delta^{\mu\nu} T_{\nu\lambda} u^\lambda$ **heat flux current (flow of energy relative to u^μ),** $q^\mu u_\mu = 0$

   $\pi^{\mu\nu} \equiv T^{<\mu\nu>}$ **shear stress tensor,** $\pi^{\mu\nu} u_\mu = \pi^{\mu\nu} u_\nu = 0\,, \ \pi^\mu{}_\mu = 0$

   $a^{(\mu\nu)} \equiv \frac{1}{2}\, (a^{\mu\nu} + a^{\nu\mu})$ **symmetrized tensor**

   $a^{<\mu\nu>} \equiv \left(\Delta_\alpha^{(\mu} \Delta^{\nu)}{}_\beta - \frac{1}{3}\, \Delta^{\mu\nu} \Delta_{\alpha\beta} \right) a^{\alpha\beta}$ **symmetrized, traceless spatial projection**

Fluid dynamics: equations of motion

1. **Net charge (e.g., strangeness) conservation:**

$$\boxed{\partial_\mu N^\mu = 0} \Longleftrightarrow \boxed{\dot{n} + n\,\theta + \partial \cdot n = 0}$$

$\dot{a} \equiv u^\mu \partial_\mu a$ convective (comoving) derivative
(time derivative in fluid rest frame, $\dot{a}_{\mathrm{RF}} \equiv \partial_t a$)

$\theta \equiv \partial_\mu u^\mu$ expansion scalar

2. **Energy-momentum conservation:**

$\boxed{\partial_\mu T^{\mu\nu} = 0} \Longleftrightarrow$ energy conservation:

$$\boxed{u_\nu\, \partial_\mu T^{\mu\nu} = \dot{\epsilon} + (\epsilon + p + \Pi)\,\theta + \partial \cdot q - q \cdot \dot{u} - \pi^{\mu\nu}\,\partial_\mu u_\nu = 0}$$

acceleration equation:

$$\boxed{\begin{array}{l} \Delta^{\mu\nu}\,\partial^\lambda T_{\nu\lambda} = 0 \quad \Longleftrightarrow \\ (\epsilon + p)\dot{u}^\mu = \nabla^\mu (p + \Pi) - \Pi \dot{u}^\mu - \Delta^{\mu\nu}\dot{q}_\nu - q^\mu\theta - q \cdot \partial u^\mu - \Delta^{\mu\nu}\,\partial^\lambda \pi_{\nu\lambda} \end{array}}$$

$\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ 3-gradient,
(spatial gradient in fluid rest frame, $u^\mu_{\mathrm{RF}} \equiv (1,0,0,0)$)

Solvability

Problem:

5 equations, but 15 unknowns (for given u^μ): ϵ , p , n , Π , n^μ (3) , q^μ (3) , $\pi^{\mu\nu}$ (5)

Solution:

1. **clever choice of frame (Eckart, Landau,...): eliminate n^μ or q^μ**

   $\Longrightarrow$ **does not help! Promotes u^μ to dynamical variable!**

2. **ideal fluid limit: all dissipative terms vanish, $\Pi = n^\mu = q^\mu = \pi^{\mu\nu} = 0$**

   $\Longrightarrow$ **6 unknowns: ϵ , p , n , u^μ (3)   (not quite there yet...)**

   $\Longrightarrow$ **fluid is in local thermodynamical equilibrium**

   $\Longrightarrow$ **provide equation of state (EOS) $p(\epsilon, n)$ to close system of equations**

3. **provide additional equations for dissipative quantities**

   $\Longrightarrow$ **dissipative relativistic fluid dynamics**

   (a) **First-order theories: e.g. generalization of Navier-Stokes (NS) equations to the relativistic case (Landau, Lifshitz)**

   (b) **Second-order theories: e.g. Israel-Stewart (IS) equations**

Navier-Stokes equations

Navier-Stokes (NS) equations: first-order, dissipative relativistic fluid dynamics

1. **bulk viscous pressure:** $\boxed{\Pi_{\rm NS} = -\zeta\,\theta}$

   ζ bulk viscosity

2. **diffusion current:** $\boxed{n^\mu_{\rm NS} = \kappa_n\,\nabla^\mu\alpha}$

   $\beta \equiv 1/T$ inverse temperature,

   $\alpha \equiv \beta\,\mu$, μ chemical potential,

   κ_n net-charge diffusion coefficient

3. **shear stress tensor:** $\boxed{\pi^{\mu\nu}_{\rm NS} = 2\,\eta\,\sigma^{\mu\nu}}$

   η shear viscosity,

   $\sigma^{\mu\nu} = \nabla^{<\mu}u^{\nu>}$ shear tensor

$\Longrightarrow$ algebraic expressions in terms of thermodynamic and fluid variables

$\Longrightarrow$ simple... but: unstable and acausal equations of motion!!

W.A. Hiscock, L. Lindblom, PRD 31 (1985) 725

Israel-Stewart equations

Israel-Stewart (IS) equations: second-order, dissipative relativistic fluid dynamics

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

"Simplified" version:

$$\boxed{\begin{aligned} \tau_\Pi\, \dot{\Pi} + \Pi &= \Pi_{\rm NS} \\ \tau_n\, \dot{n}^{<\mu>} + n^\mu &= n^\mu_{\rm NS} \\ \tau_\pi\, \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} &= \pi^{\mu\nu}_{\rm NS} \end{aligned}}$$

cf. also T. Koide, G.S. Denicol, Ph. Mota, T. Kodama, Phys. Rev. C75 (2007) 034909

$\Longrightarrow$ dynamical (instead of algebraic) equations for dissipative terms!

solution: e.g. bulk viscous pressure $\boxed{\Pi(t) = \Pi_{\rm NS}\left(1 - e^{-t/\tau_\Pi}\right) + \Pi(0)\, e^{-t/\tau_\Pi}}$

$\Longrightarrow$ dissipative quantities $\Pi\,,\ n^\mu\,,\ \pi^{\mu\nu}$ relax to their respective NS values $\Pi_{\rm NS}\,,\ n^\mu_{\rm NS}\,,\ \pi^{\mu\nu}_{\rm NS}$ on time scales $\tau_\Pi\,,\ \tau_n\,,\ \tau_\pi$

$\Longrightarrow$ stable and causal fluid dynamical equations of motion!

see, e.g., S. Pu, T. Koide, DHR, PRD81 (2010) 114039

Power counting (I)

3 length scales: 2 microscopic, 1 macroscopic

- thermal wavelength $\lambda_{\text{th}} \sim \beta \equiv 1/T$
- mean free path $\ell_{\text{mfp}} \sim (\langle\sigma\rangle n)^{-1}$

  $\langle\sigma\rangle$ averaged cross section, $n \sim T^3 = \beta^{-3} \sim \lambda_{\text{th}}^{-3}$
- length scale over which macroscopic fluid fields vary L_{hydro} , $\partial_\mu \sim L_{\text{hydro}}^{-1}$

Note: since $\eta \sim (\langle\sigma\rangle\lambda_{\text{th}})^{-1}$ $\Longrightarrow$ $\boxed{\dfrac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \dfrac{1}{\langle\sigma\rangle n}\,\dfrac{1}{\lambda_{\text{th}}} \sim \dfrac{\lambda_{\text{th}}^3}{\langle\sigma\rangle\lambda_{\text{th}}} \sim \dfrac{\lambda_{\text{th}}^3}{\langle\sigma\rangle\lambda_{\text{th}}} \sim \dfrac{\eta}{s}}$

s entropy density, $s \sim n \sim T^3 = \beta^{-3} \sim \lambda_{\text{th}}^{-3}$

$\Longrightarrow$ $\dfrac{\eta}{s}$ solely determined by the 2 microscopic length scales!

Note: similar argument holds for $\dfrac{\zeta}{s}$, $\dfrac{\kappa_n}{\beta\, s}$

Power counting (II)

3 regimes:

- dilute gas limit $\dfrac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \dfrac{\eta}{s} \gg 1 \iff \langle \sigma \rangle \ll \lambda_{\text{th}}^2 \implies$ weak-coupling limit
- viscous fluids $\dfrac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \dfrac{\eta}{s} \sim 1 \iff \langle \sigma \rangle \sim \lambda_{\text{th}}^2$

  interactions happen on the scale $\lambda_{\text{th}} \implies$ moderate coupling
- ideal fluid limit $\dfrac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \dfrac{\eta}{s} \ll 1 \iff \langle \sigma \rangle \gg \lambda_{\text{th}}^2 \implies$ strong-coupling limit

gradient (derivative) expansion: $\boxed{\ell_{\text{mfp}}\, \partial_\mu \sim \dfrac{\ell_{\text{mfp}}}{L_{\text{hydro}}} \equiv K \sim \delta \ll 1}$

K Knudsen number

$\implies$ equivalent to $k\, \ell_{\text{mfp}} \ll 1\,,\quad k$ typical momentum scale

R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100

$\implies$ separation of macroscopic fluid dynamics (large scale $\sim L_{\text{hydro}}$)

from microscopic particle dynamics (small scale $\sim \ell_{\text{mfp}}$)

Power counting (III)

Primary quantities: $\epsilon\,,\ p\,,\ n\,,\ s \quad \Longleftrightarrow \quad$ **Dissipative quantities:** $\Pi\,,\ n^\mu\,,\ \pi^{\mu\nu}$

$$\boxed{\text{If}\quad K \sim \ell_{\text{mfp}}\,\partial_\mu \sim \delta \ll 1\ ,\ \text{then}\quad \frac{\Pi}{\epsilon} \sim \frac{n^\mu}{s} \sim \frac{\pi^{\mu\nu}}{\epsilon} \sim \delta \ll 1}$$

Dissipative quantities are small compared to primary quantities

$\Longrightarrow$ **small deviations from local thermodynamical equilibrium!**

Note: statement independent of value of $\dfrac{\zeta}{s}\,,\ \dfrac{\kappa_n}{\beta\, s}\,,\ \dfrac{\eta}{s}$ **!**

Proof: **Gibbs relation:** $\epsilon + p = Ts + \mu n \quad \Longrightarrow \quad \beta\,\epsilon \sim s$ **!**

Estimate dissipative terms by their Navier-Stokes values:

$$\Pi \sim \Pi_{\text{NS}} = -\zeta\,\theta\,, \quad n^\mu \sim n^\mu_{\text{NS}} = \kappa_n\,\nabla^\mu\alpha\,, \quad \pi^{\mu\nu} \sim \pi^{\mu\nu}_{\text{NS}} = 2\,\eta\,\sigma^{\mu\nu}$$

$$\Longrightarrow \frac{\Pi}{\epsilon} \sim -\frac{\zeta}{\beta\,\epsilon}\,\beta\,\theta \sim -\frac{\zeta}{s}\,\frac{\beta}{\lambda_{\text{th}}}\,\frac{\lambda_{\text{th}}}{\ell_{\text{mfp}}}\,\ell_{\text{mfp}}\,\theta \sim \ell_{\text{mfp}}\,\partial_\mu u^\mu \sim \delta\,,$$

$$\frac{n^\mu}{s} \sim \frac{\kappa_n}{s}\,\nabla^\mu\alpha \sim \frac{\kappa_n}{\beta\, s}\,\frac{\beta}{\lambda_{\text{th}}}\,\frac{\lambda_{\text{th}}}{\ell_{\text{mfp}}}\,\ell_{\text{mfp}}\,\nabla^\mu\alpha \sim \ell_{\text{mfp}}\,\nabla^\mu\alpha \sim \delta\,,$$

$$\frac{\pi^{\mu\nu}}{\epsilon} \sim 2\,\frac{\eta}{\beta\,\epsilon}\,\beta\,\sigma^{\mu\nu} \sim 2\,\frac{\eta}{s}\,\frac{\beta}{\lambda_{\text{th}}}\,\frac{\lambda_{\text{th}}}{\ell_{\text{mfp}}}\,\ell_{\text{mfp}}\,\sigma^{\mu\nu} \sim \ell_{\text{mfp}}\,\nabla^{<\mu}u^{\nu>} \sim \delta\ , \qquad \text{q.e.d.}$$

Israel-Stewart equations revisited (I)

additional relaxation term in IS equation is of second order in δ:

$$\frac{1}{\epsilon}\,\tau_\Pi\,\dot{\Pi} \sim \frac{1}{\epsilon}\,u^\mu\,\ell_{\rm mfp}\,\partial_\mu \Pi \sim K\,\frac{\Pi}{\epsilon} \sim K\,\delta \sim O(\delta^2)$$

$\Longrightarrow$ to be consistent, have to include other second-order terms as well!

$$\boxed{\begin{aligned} \tau_\Pi\,\dot{\Pi} + \Pi &= \Pi_{\rm NS} + \mathcal{K} \\ \tau_n\,\dot{n}^{<\mu>} + n^\mu &= n^\mu_{\rm NS} + \mathcal{K}^\mu \\ \tau_\pi\,\dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} &= \pi^{\mu\nu}_{\rm NS} + \mathcal{K}^{\mu\nu} \end{aligned}}$$

$$\mathcal{K} = \zeta_1\,\omega_{\mu\nu}\,\omega^{\mu\nu} + \zeta_2\,\sigma^{\mu\nu}\,\sigma_{\mu\nu} + \zeta_3\,\theta^2 + \zeta_4\,(\nabla\alpha)^2 + \zeta_5\,(\nabla p)^2 + \zeta_6\,\nabla\alpha\cdot\nabla p + \zeta_7\,\nabla^2\alpha + \zeta_8\,\nabla^2 p\,,$$

$$\mathcal{K}^\mu = \kappa_1\,\sigma^{\mu\nu}\,\nabla_\nu\alpha + \kappa_2\,\sigma^{\mu\nu}\,\nabla_\nu p + \kappa_3\,\theta\,\nabla^\mu\alpha + \kappa_4\,\theta\,\nabla^\mu p + \kappa_5\,\omega^{\mu\nu}\,\nabla_\nu\alpha + \kappa_6\,\Delta^{\mu\lambda}\nabla^\nu\sigma_{\lambda\nu} + \kappa_7\,\nabla^\mu\theta\,,$$

$$\begin{aligned}\mathcal{K}^{\mu\nu} = {} & \eta_1\,\omega_\lambda{}^{<\mu}\,\omega^{\nu>\lambda} + \eta_2\,\theta\,\sigma^{\mu\nu} + \eta_3\,\sigma_\lambda{}^{<\mu}\,\sigma^{\nu>\lambda} + \eta_4\,\sigma_\lambda{}^{<\mu}\,\omega^{\nu>\lambda} + \eta_5\,\nabla^{<\mu}\alpha\,\nabla^{\nu>}\alpha \\ & + \eta_6\,\nabla^{<\mu}p\,\nabla^{\nu>}p + \eta_7\,\nabla^{<\mu}\alpha\,\nabla^{\nu>}p + \eta_8\,\nabla^{<\mu}\nabla^{\nu>}\alpha + \eta_9\,\nabla^{<\mu}\nabla^{\nu>}p\end{aligned}$$

where $\omega^{\mu\nu} \equiv \nabla^{<\mu}u^{\nu>}$ vorticity

$\Longrightarrow$ second-order gradient expansion!

cf. R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100
P. Romatschke, Class. Quant. Grav. 27 (2010) 025006

Israel-Stewart equations revisited (II)

unfortunately, including second-order gradient terms renders eqs. of motion parabolic

$\Longrightarrow$ acausal, unstable $\Longrightarrow$ in general, $\mathcal{K}$, $\mathcal{K}^\mu$, $\mathcal{K}^{\mu\nu}$ have to be omitted!

. . . but there is more: in principle, Π , n^μ , $\pi^{\mu\nu}$ are quantities independent from

$$\theta\,,\ \nabla^\mu\alpha\,,\ \nabla^\mu p\,,\ \sigma^{\mu\nu}\,,\ \omega^{\mu\nu}$$

$\Longrightarrow$ additional Lorentz-covariants:

$$\boxed{\begin{aligned}
\tau_\Pi\,\dot{\Pi} + \Pi &= \Pi_{\rm NS} + \mathcal{K} + \mathcal{J} + \mathcal{R}\\
\tau_n\,\dot{n}^{<\mu>} + n^\mu &= n^\mu_{\rm NS} + \mathcal{K}^\mu + \mathcal{J}^\mu + \mathcal{R}^\mu\\
\tau_\pi\,\dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} &= \pi^{\mu\nu}_{\rm NS} + \mathcal{K}^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu}
\end{aligned}}$$

$$\begin{aligned}
\mathcal{J} &= -\ell_{\Pi n}\,\nabla\cdot n - \tau_{\Pi n}\,n\cdot\nabla p - \delta_{\Pi\Pi}\,\theta\,\Pi - \lambda_{\Pi n}\,n\cdot\nabla\alpha + \lambda_{\Pi\pi}\,\pi^{\mu\nu}\sigma_{\mu\nu}\\
\mathcal{J}^\mu &= \omega^{\mu\nu}\,n_\nu - \delta_{nn}\,\theta\,n^\mu - \ell_{n\Pi}\,\nabla^\mu\Pi + \ell_{n\pi}\Delta^{\mu\nu}\,\nabla^\lambda\pi_{\nu\lambda} + \tau_{n\Pi}\,\Pi\,\nabla^\mu p - \tau_{n\pi}\,\pi^{\mu\nu}\,\nabla_\nu p - \lambda_{nn}\,\sigma^{\mu\nu}\,n_\nu + \lambda_{n\Pi}\,\Pi\,\nabla^\mu\alpha\\
&\quad - \lambda_{n\pi}\,\pi^{\mu\nu}\,\nabla_\nu\alpha\\
\mathcal{J}^{\mu\nu} &= 2\,\pi_\lambda{}^{<\mu}\,\omega^{\nu>\lambda} - \delta_{\pi\pi}\,\theta\,\pi^{\mu\nu} - \tau_{\pi\pi}\,\pi_\lambda{}^{<\mu}\,\sigma^{\nu>\lambda} + \lambda_{\pi\Pi}\,\Pi\,\sigma^{\mu\nu} - \tau_{\pi n}\,n^{<\mu}\,\nabla^{\nu>}p + \ell_{\pi n}\,\nabla^{<\mu}n^{\nu>} + \lambda_{\pi n}\,n^{<\mu}\nabla^{\nu>}\alpha\\
\mathcal{R} &= \varphi_1\,\Pi^2 + \varphi_2\,n\cdot n + \varphi_3\,\pi^{\mu\nu}\pi_{\mu\nu}\\
\mathcal{R}^\mu &= \varphi_4\,\pi^{\mu\nu}\,n_\nu + \varphi_5\,\Pi\,n^\mu\\
\mathcal{R}^{\mu\nu} &= \varphi_6\,\Pi\,\pi^{\mu\nu} + \varphi_7\,\pi_\lambda{}^{<\mu}\,\pi^{\nu>\lambda} + \varphi_8\,n^{<\mu}\,n^{\nu>}
\end{aligned}$$

Matching to kinetic theory

Fluid dynamics is an effective (macroscopic) theory for the long-wavelength, small-frequency limit of a given (microscopic) theory

$\Longrightarrow$ **coefficients in equations of motion can be determined by matching to the underlying theory, e.g. kinetic theory**

G.S. Denicol, H. Niemi, E. Molnár, DHR, PRD85 (2012) 114047

Details (I)

1. Boltzmann eq. $k \cdot \partial f_k = C[f]$ for single-particle distribution function f_k
2. introduce irreducible tensors of rank ℓ: $k_{<\mu_1} \cdots k_{\mu_\ell>} \equiv \Delta^{\nu_1 \cdots \nu_\ell}_{\mu_1 \cdots \mu_\ell} k_{\nu_1} \cdots k_{\nu_\ell}$

   $\Delta^{\nu_1 \cdots \nu_\ell}_{\mu_1 \cdots \mu_\ell}$ are projectors onto subspaces orthogonal to u^μ, symmetric in μ_i, ν_i, and traceless
3. f_k can be expanded in terms of $k_{<\mu_1} \cdots k_{\mu_\ell>}$

$$\boxed{f_k = f_{0k} + f_{0k}\, \tilde{f}_{0k} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}^{(\ell)}_{kn}\, \rho_n^{\mu_1 \cdots \mu_\ell}\, k_{<\mu_1} \cdots k_{\mu_\ell>}}$$

where

(a) $f_{0k} = [\exp(\beta\, u \cdot k - \alpha) + a]^{-1}$ single-particle distribution function in local equilibrium, $a = \pm 1/0$ for Fermi/Bose/Boltzmann statistics

(b) $\tilde{f}_{0k} = 1 - a\, f_{0k}$

(c) $\mathcal{H}^{(\ell)}_{kn} = \dfrac{W^{(\ell)}}{\ell!} \sum_{m=n}^{N_\ell} a^{(\ell)}_{mn}\, P^{(\ell)}_{km}$, where

$P^{(\ell)}_{kn} = \sum_{r=0}^{n} a^{(\ell)}_{nr}\, E_k^r$ are orthogonal polynomials of order n in energy $E_k \equiv u \cdot k$

$\Longrightarrow$ $\mathcal{H}^{(\ell)}_{kn}$ are polynomials of order N_ℓ in energy E_k

(d) irreducible moments of $\delta f_k \equiv f_k - f_{0k}$: $\rho_n^{\mu_1 \cdots \mu_\ell} = \int \mathrm{d}K\, \delta f_k\, E_k^n\, k^{<\mu_1} \cdots k^{\mu_\ell>}$

Details (II)

4. rewrite Boltzmann equation in the form $\delta \dot{f}_k = -\dot{f}_{0k} - \frac{1}{E_k}\left\{k\cdot\nabla\left(f_{0k}+\delta f_k\right) - C[f]\right\}$

5. derive equations of motion for irreducible moments, e.g. up to $\ell = 2$:

$$\begin{aligned}
\dot{\rho}_r &= C_{r-1} + \alpha_r^{(0)}\,\theta - \tfrac{G_{2r}}{D_{20}}\,\theta\,\Pi + \tfrac{G_{2r}}{D_{20}}\,\sigma^{\mu\nu}\,\pi_{\mu\nu} + \tfrac{G_{3r}}{D_{20}}\,\partial\cdot n + (r-1)\sigma_{\mu\nu}\,\rho^{\mu\nu}_{r-2} + r\dot{u}_\mu\rho^\mu_{r-1} \\
&\quad - \nabla_\mu\rho^\mu_{r-1} - \tfrac{1}{3}\left[(r+2)\rho_r - (r-1)m^2\rho_{r-2}\right]\theta \\
\dot{\rho}_r^{<\mu>} &= C_{r-1}^{<\mu>} + \alpha_r^{(1)}\,\nabla^\mu\alpha + \omega^\mu{}_\nu\rho^\nu_r - \tfrac{1}{3}\left[(r+3)\rho^\mu_r - (r-1)m^2\rho^\mu_{r-2}\right]\theta - \Delta^\mu_\lambda\nabla_\nu\rho^{\lambda\nu}_{r-1} \\
&\quad - \tfrac{1}{5}\left[(2r+3)\rho^\nu_r - 2(r-1)m^2\rho^\nu_{r-2}\right]\sigma^\mu_\nu - \tfrac{1}{3}\left[(r+3)\rho_{r+1} - rm^2\rho_{r-1}\right]\dot{u}^\mu \\
&\quad + \tfrac{\beta J_{r+2,1}}{\epsilon+p}\left(\Pi\dot{u}^\mu - \nabla^\mu\Pi + \Delta^{\mu\nu}\partial^\lambda\pi_{\lambda\nu}\right) + \tfrac{1}{3}\nabla^\mu\left(\rho_{r+1} - m^2\rho_{r-1}\right) \\
&\quad + (r-1)\rho^{\mu\nu\lambda}_{r-2}\sigma_{\lambda\nu} + r\dot{u}_\nu\rho^{\mu\nu}_{r-1} \\
\dot{\rho}_r^{<\mu\nu>} &= C_{r-1}^{<\mu\nu>} + 2\,\alpha_r^{(2)}\sigma^{\mu\nu} - \tfrac{2}{7}\left[(2r+5)\rho_r^{\lambda<\mu} - 2(r-1)m^2\rho_{r-2}^{\lambda<\mu}\right]\sigma_\lambda^{\nu>} + 2\rho_r^{\lambda<\mu}\omega_\lambda{}^{\nu>} \\
&\quad + \tfrac{2}{15}\left[(r+4)\rho_{r+2} - (2r+3)m^2\rho_r + (r-1)m^4\rho_{r-2}\right]\sigma^{\mu\nu} \\
&\quad + \tfrac{2}{5}\nabla^{<\mu}\left(\rho^{\nu>}_{r+1} - m^2\rho^{\nu>}_{r-2}\right) - \tfrac{2}{5}\left[(r+5)\rho^{<\mu}_{r+1} - rm^2\rho^{<\mu}_{r-1}\right]\dot{u}^{\nu>} \\
&\quad - \tfrac{1}{3}\left[(r+4)\rho^{\mu\nu}_r - (r-1)m^2\rho^{\mu\nu}_{r-2}\right]\theta + (r-1)\rho^{\mu\nu\lambda\rho}_{r-2}\sigma_{\lambda\rho} - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda\rho^{\alpha\beta\lambda}_{r-1} + r\rho^{\mu\nu\lambda}_{r-1}\dot{u}_\lambda
\end{aligned}$$

$\alpha_r^{(\ell)}$, G_{nm}, D_{nq}, J_{nq} thermodynamic functions

$C_r^{<\mu_1\cdots\mu_\ell>} = \int \mathrm{d}K\, E_k^r\, k^{<\mu_1}\cdots k^{\mu_\ell>}\, C[f]$ irreducible moment of collision integral

Details (III)

Remarks:

(a) system of infinitely many coupled equations for irreducible moments $\rho_r^{\mu_1\cdots\mu_\ell}$

(b) system completely equivalent to Boltzmann equation

(c) by definition $\rho_0 = -\dfrac{3}{m^2}\Pi\,,\ \rho_0^\mu = n^\mu\,,\ \rho_0^{\mu\nu} = \pi^{\mu\nu}$

(d) matching conditions in Landau frame imply $\rho_1 = \rho_2 = \rho_1^\mu = 0$

6. fluid dynamics comprises tensors up to rank 2 $\Longrightarrow$ neglect $\rho_r^{\mu_1\cdots\mu_\ell}$ with $\ell > 2$

7. linearize collision integral: $C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\sum_{n=0}^{N_\ell} \mathcal{A}_{rn}^{(\ell)}\,\rho_n^{\mu_1\cdots\mu_\ell} + O(\delta f_k^2)$

$\Longrightarrow$ linearized equation of motion for irreducible moments:

$$\boxed{\begin{aligned} \dot{\vec{\rho}} + \mathcal{A}^{(0)}\,\vec{\rho} &\simeq \vec{\alpha}^{(0)}\theta + \ldots \\ \dot{\vec{\rho}}^{\,\mu} + \mathcal{A}^{(1)}\,\vec{\rho}^{\,\mu} &\simeq \vec{\alpha}^{(1)}\nabla^\mu\alpha + \ldots \\ \dot{\vec{\rho}}^{\,\mu\nu} + \mathcal{A}^{(2)}\,\vec{\rho}^{\,\mu\nu} &\simeq 2\,\vec{\alpha}^{(2)}\sigma^{\mu\nu} + \ldots \end{aligned}}$$

8. diagonalize collision matrix: $(\Omega^{-1})^{(\ell)}\mathcal{A}^{(\ell)}\Omega^{(\ell)} = \mathrm{diag}(\chi_0^{(\ell)},\ldots,\chi_j^{(\ell)},\ldots)$

for later purposes: $\tau^{(\ell)} \equiv \left(\mathcal{A}^{-1}\right)^{(\ell)} = \Omega^{(\ell)}\ \mathrm{diag}\left(1/\chi_0^{(\ell)}\,,\ \ldots\,,\ 1/\chi_j^{(\ell)}\,,\ \ldots\right)\ \left(\Omega^{-1}\right)^{(\ell)}$

$$\Longrightarrow \quad \boxed{\begin{aligned} \tau^{(0)}\,\dot{\vec{\rho}} + \vec{\rho} &\simeq \tau^{(0)}\,\vec{\alpha}^{(0)}\theta + \ldots \\ \tau^{(1)}\,\dot{\vec{\rho}}^{\,\mu} + \vec{\rho}^{\,\mu} &\simeq \tau^{(1)}\,\vec{\alpha}^{(1)}\nabla^\mu\alpha + \ldots \\ \tau^{(2)}\,\dot{\vec{\rho}}^{\,\mu\nu} + \vec{\rho}^{\,\mu\nu} &\simeq 2\,\tau^{(2)}\,\vec{\alpha}^{(2)}\sigma^{\mu\nu} + \ldots \end{aligned}}$$

Details (IV)

9. eigenmodes of linearized equations of motion: $\boldsymbol{X}_i^{\mu_1\cdots\mu_\ell} = \sum_{j=0}^{N_\ell} (\Omega^{-1})_{ij}^{(\ell)} \rho_j^{\mu_1\cdots\mu_\ell}$

$\Longrightarrow$ equations of motion for eigenmodes decouple:

$$\boxed{\begin{aligned} \dot{X}_i + \chi_i^{(0)} X_i &= \beta_i^{(0)} \theta + \ldots \\ \dot{X}_i^{<\mu>} + \chi_i^{(1)} X_i^\mu &= \beta_i^{(1)} \nabla^\mu \alpha + \ldots \\ \dot{X}_i^{<\mu\nu>} + \chi_i^{(2)} X_i^{\mu\nu} &= \beta_i^{(2)} \sigma^{\mu\nu} + \ldots \end{aligned}}$$

where $\beta_i^{(0)} = \sum_{j=0,\neq 1,2}^{N_0} \left(\Omega^{-1}\right)_{ij}^{(0)} \alpha_j^{(0)}\,, \quad \beta_i^{(1)} = \sum_{j=0,\neq 1}^{N_1} \left(\Omega^{-1}\right)_{ij}^{(1)} \alpha_j^{(1)}\,, \quad \beta_i^{(2)} = 2 \sum_{j=0}^{N_2} \left(\Omega^{-1}\right)_{ij}^{(2)} \alpha_j^{(2)}$

10. slowest eigenmodes (w/o r.o.g. $i = 0$) remain dynamical,

all faster ones ($i \neq 0$) are replaced by their asymptotic (NS) values:

$$X_i \simeq \frac{\beta_i^{(0)}}{\chi_i^{(0)}} \theta\,, \quad X_i^\mu \simeq \frac{\beta_i^{(1)}}{\chi_i^{(1)}} \nabla^\mu \alpha\,, \quad X_i^{\mu\nu} \simeq \frac{\beta_i^{(2)}}{\chi_i^{(2)}} \sigma^{\mu\nu}$$

11. Since $\rho_i^{\mu_1\cdots\mu_\ell} = \sum_{j=0}^{N_\ell} \Omega_{ij}^{(\ell)} X_j^{\mu_1\cdots\mu_\ell}$:

$$\begin{aligned} \rho_i &\simeq \Omega_{i0}^{(0)} X_0 + \sum_{j=3}^{N_0} \Omega_{ij}^{(0)} \frac{\beta_j^{(0)}}{\chi_j^{(0)}} \theta \\ \rho_i^\mu &\simeq \Omega_{i0}^{(1)} X_0^\mu + \sum_{j=2}^{N_1} \Omega_{ij}^{(1)} \frac{\beta_j^{(1)}}{\chi_j^{(1)}} \nabla^\mu \alpha \\ \rho_i^{\mu\nu} &\simeq \Omega_{i0}^{(2)} X_0^{\mu\nu} + \sum_{j=1}^{N_2} \Omega_{ij}^{(2)} \frac{\beta_i^{(2)}}{\chi_j^{(2)}} \sigma^{\mu\nu} \end{aligned}$$

Details (V)

$\Longrightarrow$ express X_0 , X_0^μ , $X_0^{\mu\nu}$ in terms of Π , n^μ , $\pi^{\mu\nu}$ as well as θ , $\nabla^\mu\alpha$, $\sigma^{\mu\nu}$

(w/o r.o.g. $\Omega_{00}^{(\ell)} \equiv 1$): $X_0 \simeq -\frac{3}{m^2}\Pi - \sum_{j=3}^{N_0} \Omega_{0j}^{(0)} \frac{\beta_j^{(0)}}{\chi_j^{(0)}} \theta$

$$X_0^\mu \simeq n^\mu - \sum_{j=2}^{N_1} \Omega_{0j}^{(1)} \frac{\beta_j^{(1)}}{\chi_j^{(1)}} \nabla^\mu\alpha$$

$$X_0^{\mu\nu} \simeq \pi^{\mu\nu} - \sum_{j=1}^{N_2} \Omega_{0j}^{(2)} \frac{\beta_i^{(2)}}{\chi_j^{(2)}} \sigma^{\mu\nu}$$

$\Longrightarrow$ express ρ_i , ρ_i^μ , $\rho_i^{\mu\nu}$ in terms of Π , n^μ , $\pi^{\mu\nu}$ as well as θ , $\nabla^\mu\alpha$, $\sigma^{\mu\nu}$:

$$\boxed{\begin{aligned} \frac{m^2}{3}\rho_i &\simeq -\Omega_{i0}^{(0)}\Pi + \left(\zeta_i - \Omega_{i0}^{(0)}\zeta_0\right)\theta \\ \rho_i^\mu &\simeq \Omega_{i0}^{(1)} n^\mu + \left(\kappa_{n\,i} - \Omega_{i0}^{(1)}\kappa_{n\,0}\right)\nabla^\mu\alpha \\ \rho_i^{\mu\nu} &\simeq \Omega_{i0}^{(2)}\pi^{\mu\nu} + 2\left(\eta_i - \Omega_{i0}^{(2)}\eta_0\right)\sigma^{\mu\nu} \end{aligned}}$$

where $\zeta_i = \frac{m^2}{3} \sum_{r=0,\neq 1,2}^{N_0} \tau_{ir}^{(0)}\alpha_r^{(0)}$, $\kappa_{n\,i} = \sum_{r=0,\neq 1}^{N_1} \tau_{ir}^{(1)}\alpha_r^{(1)}$, $\eta_i = \sum_{r=0}^{N_2} \tau_{ir}^{(2)}\alpha_r^{(2)}$

$\Longrightarrow$ equations of motion for irreducible moments become identical with equations of motion for dissipative quantities Π , n^μ , $\pi^{\mu\nu}$

$\Longrightarrow$ identify transport coefficients

Discussion (I)

1. **Basis of expansion for δf_k is orthogonal in irreducible subspaces**
   $\Longrightarrow$ **truncation at any order in ℓ and N_ℓ possible!**
2. **14-moment approximation corresponds to choice $N_0 = 2\,,\ N_1 = 1\,,\ N_2 = 0$ and leads to IS equations**
3. **approximation can be systematically improved by increasing N_ℓ**
4. **transport coefficients approach Chapman-Enskog values already for $N_0 = 5\,,\ N_1 = 4\,,\ N_2 = 3$ (41-moment approximation)**
   Example: classical massless gas with constant cross section σ, $\ell_{\rm mfp} = (\sigma n)^{-1}$

# of moments	η	$\tau_\pi[\ell_{\rm mfp}]$	$\tau_{\pi\pi}[\tau_\pi]$	$\lambda_{\pi n}[\tau_\pi]$	$\delta_{\pi\pi}[\tau_\pi]$	$\ell_{\pi n}[\tau_\pi]$	$\tau_{\pi n}[\tau_\pi]$
14	$4/(3\sigma\beta)$	$5/3$	$10/7$	0	$4/3$	0	0
23	$14/(11\sigma\beta)$	2	$134/77$	$0.344/\beta$	$4/3$	$-0.689/\beta$	$-0.689/n$
32	$1.268/(\sigma\beta)$	2	1.69	$0.254/\beta$	$4/3$	$-0.687/\beta$	$-0.687/n$
41	$1.267/(\sigma\beta)$	2	1.69	$0.244/\beta$	$4/3$	$-0.685/\beta$	$-0.685/n$

# of moments	κ	$\tau_n[\ell_{\rm mfp}]$	$\delta_{nn}[\tau_n]$	$\lambda_{nn}[\tau_n]$	$\lambda_{n\pi}[\tau_n]$	$\ell_{n\pi}[\tau_n]$	$\tau_{n\pi}[\tau_n]$
14	$3/\,(16\sigma)$	$9/4$	1	$3/5$	$\beta/20$	$\beta/20$	$0.0125\beta/p$
23	$21/\,(128\sigma)$	2.59	1.0	0.96	0.054β	0.118β	$0.0295\beta/p$
32	$0.1605/\sigma$	2.57	1.0	0.93	0.052β	0.119β	$0.0297\beta/p$
41	$0.1596/\sigma$	2.57	1.0	0.92	0.052β	0.119β	$0.0297\beta/p$

Discussion (II)

5. approach can be further systematically improved:
   (a) consider also faster eigenmodes $X_i\,,\ X_i^{\mu}\,,\ X_i^{\mu\nu}\,,\ i>0$, to be dynamical
   (b) take into account irreducible moments of tensor rank $\ell > 2$
   (c) take into account second-order corrections in the collision integral (compute coefficients $\varphi_1, \ldots, \varphi_8$)

   E. Molnar, H. Niemi, G.S. Denicol, DHR, arXiv:1308.0785 [nucl-th]

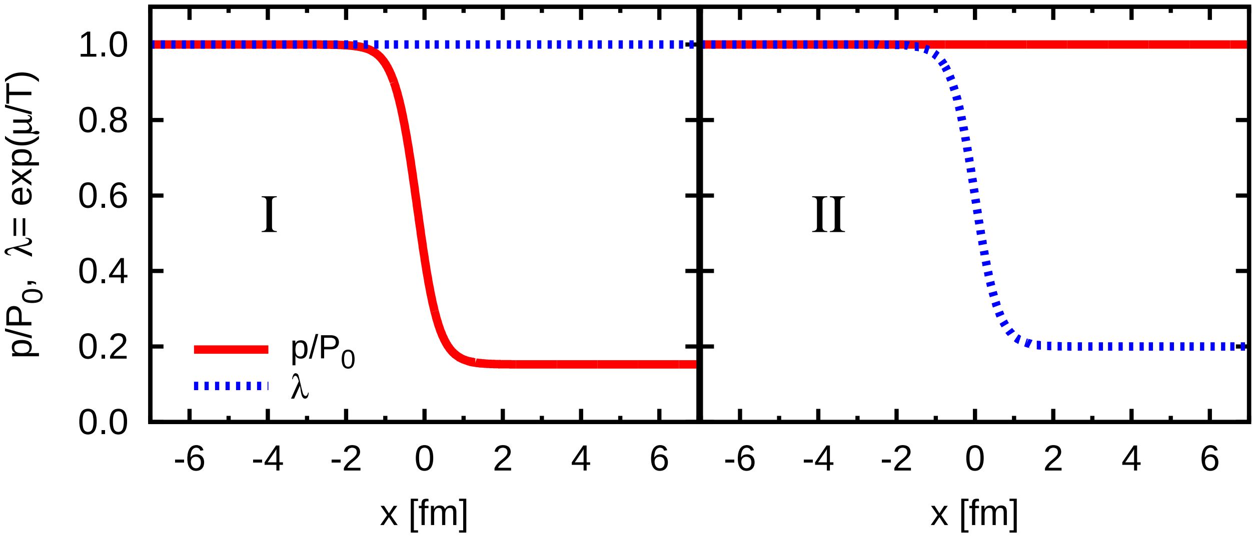
   e.g. massless Boltzmann gas: $\varphi_4 = \dfrac{1}{25\,p}\,,\ \varphi_7 = \dfrac{9}{70\,p}\,,\ \varphi_8 = \dfrac{8}{5\,\beta^2 p}$

Application: heat-flow problem (I)

G.S. Denicol, H. Niemi, I. Bouras, E. Molnár, Z. Xu, DHR, C. Greiner, arXiv:1207.6811[nucl-th]

Initial conditions: discontinuity in

(I) pressure (II) fugacity



$\Longrightarrow$ first-order (NS) terms can be vanishingly small:

$$\text{(I)}: \nabla^\mu \alpha \simeq 0 \qquad \text{(II)}: \nabla^\mu p \simeq \dot{u}^\mu \simeq 0 \quad \Longrightarrow \quad \sigma^{\mu\nu} \simeq 0$$

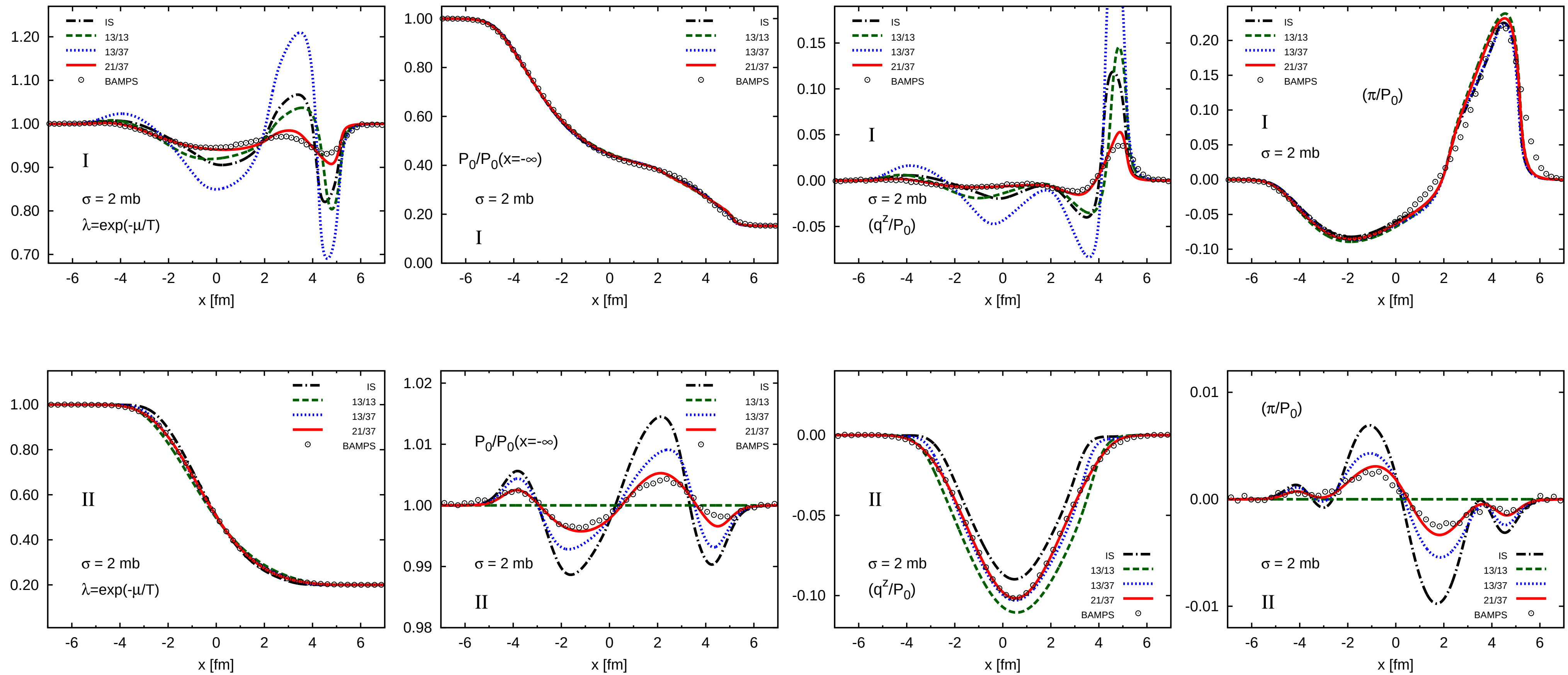
$\Longrightarrow$ second-order terms can become larger than first-order terms!

$\Longrightarrow$ power-counting scheme in terms of Knudsen number is invalidated!

Application: heat-flow problem (II)

Solution: consider ρ_2^{μ} , $\rho_1^{\mu\nu}$ as additional dynamical variables!

$e^{-\alpha}$ $\quad$ p $\quad$ q^{μ} $\quad$ $\pi^{\mu\nu}$



Conclusions

1. **Second-order fluid dynamics has been systematically derived as long-wavelength, small-frequency limit of kinetic theory**
2. **Transport coefficients agree with values from Chapman-Enskog expansion**
3. **Further systematic improvements are possible and should be explored**
4. **Heat-flow problem can be solved by taking higher irreducible moments to be dynamical variables**

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Happy Birthday, Takeshi, and many more Similar Ones!