Relativistic dissipative hydrodynamics with extended matching conditions

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1.1 Introduction: Why?

Relativistic dissipative fluid dynamical models always use the following Matching conditions:

$$\boxed{\begin{aligned} &\delta T^{\mu\nu} u_\mu u_\nu = 0 \\ &\delta n^\mu u_\mu = 0 \end{aligned}}$$

Energy density and particle number density in non-equilibrium is exactly same as those in equilibrium. *Is this correct?*

1.2 An additional motivation

There are exp. results shows non-extensive behavior in high-energy nuclear collisions. (p_{T} distribution)

There are also exp. results indicates existence of long-range correlations which brakes extensivity. (Ridge phenomena)

Non-extensive perfect fluid works nicely.
It corresponds to Extensive(Usual) dissipative fluid

T.O. and G.Wilk, PRC77, 044903 (2008), T.O. PRC81, 024907 (2010).

A difference, naturally we have $\delta T^{\mu\nu} u_\mu u_\nu \neq 0$

1.3 Kinetic approach

Working theory:

G.S. Denicol *et.al*, Phys. Rev. D85, 114047 (2012)

Relativistic Kinetic theory

using irreducible tensors $1, k^{<\mu>}, k^{<\mu}k^{\nu>}, \cdots$ base

$\Longrightarrow$ deviation (distribution function) from equilibrium

Today's talk:

+ Extended Matching conditions *[RANP2013]*

$$\delta T^{\mu\nu} u_\mu u_\nu \neq 0$$

$$\delta n^\mu u_\mu \neq 0$$

$\Rightarrow$ *off-eq. entropy current*

initial conditions of the fluid

2.0 Setting

In arbitrary fixed frame,
Consider fluid before equilibrium.

particle [constitute the fluid] momentum

$k^{\mu} = E_k u^{\mu} + k^{<\mu>}$ Energy E_k, flow velocity u^{μ}

Distribution func. $f_k(x)$ *obeys relativistic Boltzmann eq.*

$k^{\mu}\partial_{\mu} f_k(x) = C[f]$

Metric: $g^{\mu\nu}$=diag.(+1,-1,-1,-1)
Projection tensor $\Delta^{\mu\nu}$=diag.(0,-1-1-1)

2.1 Irreducible tensors

$$k^{<\mu_1} k^{\mu_2} \ldots k^{\mu_m>} \equiv \Delta^{\mu_1 \mu_2 \cdots \mu_m}_{\nu_1 \nu_2 \cdots \nu_m} \; k^{\nu_1} k^{\nu_2} \ldots k^{\nu_m}$$

(generalized) projection tensor

symmetric :exchange μ suffixes , ν suffixes

= 0 :contraction μ suffixes , ν suffixes

orthogonality:

$$\int \frac{d^3k\, F(E_k)}{(2\pi)^3 k^0} \; k^{<\mu_1} k^{\mu_2} \ldots k^{\mu_m>} \, k_{<\nu_1} k_{\nu_2} \cdots k_{\nu_n>}$$

$$= \frac{\delta^m_n \, m!}{(2m+1)!!} \int \frac{d^3k\, F(E_k)}{(2\pi)^3 k^0} \left[\Delta_{\alpha\beta} k^\alpha k^\beta\right]^m$$

= complete set =

2.2 Expansion of the *deviation of equilibrium*

$$f_k(x) = f_0(x)[\,1 + \phi(k,x)\,]$$

equilibrium *deviation*

$$\phi(k,x) = \sum_{l=0}^{\infty} \lambda_k^{\langle \mu_1 \mu_2 \cdots \mu_l \rangle} k_{<\mu_1} k_{\mu_2} \cdots k_{\mu_l>}$$

All information about deviation
⟹ Initial condition of dissipative hydro.

If all $\lambda_k^{\langle \mu_1 \mu_2 \cdots \mu_l \rangle}$ were given,
can we find the initial conditions of dissipative hydro?

2.3 Separation of k dependence from λ_k

$$\phi(k,x) = \sum_{l=0}^{\infty} \lambda_k^{\langle\mu_1\mu_2\cdots\mu_l\rangle} k_{<\mu_1} k_{\mu_2} \cdots k_{\mu_l>}$$

$$\lambda_k^{\langle\mu_1\mu_2\cdots\mu_l\rangle} = \sum_{n=0}^{N_l} c_n^{<\mu_1\mu_2\cdots\mu_l>} \; P_n^{(l)}$$

All information about off-equilib.

orthogonal Polynomial E_k

$$P_n^{(l)} = \sum_{r=0}^{n} a_{nr}^{(l)} E_k^r$$

$$\int \frac{d^3k\,\omega^{(l)}}{(2\pi)^3 k^0} P_m^{(l)} P_n^{(l)} = \delta_{mn}$$

$$\omega^{(l)} \equiv \frac{W^{(l)}}{(2l+1)!!} \left[\Delta_{\alpha\beta} k^\alpha k^\beta\right]^l f_o$$

$$1/_{W^{(l)}} \equiv (-1)^l \frac{(mT)^{l+1}}{2\pi^2} e^{\frac{\mu}{T}} K_{l+1}(m/T)$$

2.4 kinetic ⇋ fluid dynamic

$$\langle E_k^r \, k^{<\nu_1} \cdots k^{\nu_l>} \rangle_\delta \equiv \int \frac{d^3k \, \phi \, f_0}{(2\pi)^3 k_0} E^r \, k^{<\nu_1} \cdots k^{\nu_l>}$$

Using orthogonal property and solve it inversely

triangle matrix of polynm. coeffcs.

$$\begin{pmatrix} c_0^{<\mu_1\mu_2\cdots\mu_l>} \\ c_1^{<\mu_1\mu_2\cdots\mu_l>} \\ \vdots \\ c_{N_l}^{<\mu_1\mu_2\cdots\mu_l>} \end{pmatrix} = \begin{pmatrix} a_{00}^{(l)} & 0 & \cdots & 0 \\ a_{10}^{(l)} & a_{11}^{(l)} & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ a_{N_l,0}^{(l)} & a_{N_l,1}^{(l)} & \cdots & a_{N_l,N_l}^{(l)} \end{pmatrix} \begin{pmatrix} \langle (E_k)^0 k^{<\nu_1} \cdots k^{\nu_l>} \rangle_\delta \\ \langle (E_k)^1 k^{<\nu_1} \cdots k^{\nu_l>} \rangle_\delta \\ \vdots \\ \langle (E_k)^{N_l} k^{<\nu_1} \cdots k^{\nu_l>} \rangle_\delta \end{pmatrix}$$

Macro off-equilibrium

The lowest possible scheme ($l \leq 2$) $N_{l=0} = 2, N_{l=1} = 1, N_{l=2} = 0$

$$\delta T^{\mu\nu} = \underbrace{\langle E_k^2 \rangle_\delta}_{\to \Lambda} u^\mu u^\nu + \underbrace{\langle E_k k^{\langle\mu\rangle} \rangle_\delta}_{\to W^\mu} u^\nu + \underbrace{\langle E_k k^{\langle\nu\rangle} \rangle_\delta}_{\to W^\nu} u^\mu + \underbrace{\langle k^{<\mu} k^{\nu>} \rangle_\delta}_{\to \pi^{\mu\nu}} + \underbrace{\frac{1}{3} \langle \Delta_{\alpha\beta} k^\alpha k^\beta \rangle_\delta}_{\to -\Pi} \Delta^{\mu\nu}$$

$$\delta N^\mu = \underbrace{\langle E_k \rangle_\delta}_{\to \delta n} u^\mu + \underbrace{\langle k^{\langle\sigma\rangle} \rangle_\delta}_{\to V^\mu} \Delta_\sigma^\mu$$

New matching conditions $\Lambda \neq 0, \delta n \neq 0$

2.5 Matrix expression for λ_k

$$\lambda_k = \underbrace{\frac{W^{(0)}}{0!}\left(\frac{\Lambda - 3\Pi}{m^2}, \delta n\,, \Lambda\right) \times \begin{pmatrix} a_{00}^{(0)} & a_{10}^{(0)} & a_{20}^{(0)} \\ 0 & a_{11}^{(0)} & a_{21}^{(0)} \\ 0 & 0 & a_{22}^{(0)} \end{pmatrix}}_{c_n^{<\mu_1\mu_2>}} \underbrace{\begin{pmatrix} a_{00}^{(0)} & 0 & 0 \\ a_{10}^{(0)} & a_{11}^{(0)} & 0 \\ a_{20}^{(0)} & a_{21}^{(0)} & a_{22}^{(0)} \end{pmatrix} \begin{pmatrix} 1 \\ E_k \\ E_k^2 \end{pmatrix}}_{P_n^{(l)}}$$

surprisingly simple

$$\begin{pmatrix} 1 & I_1^{(l)} & I_2^{(l)} \\ I_1^{(l)} & I_2^{(l)} & I_3^{(l)} \\ I_2^{(l)} & I_3^{(l)} & I_4^{(l)} \end{pmatrix}^{-1}$$

$$I_r^{(l)} = \int \frac{d^3k\, \omega^{(l)}}{(2\pi)^3 k^0} E_k^r$$

Operate $\int \frac{d^3k\, \omega^{(l)}}{(2\pi)^3 k^0}$, one can easily extract $\frac{\Lambda - 3\Pi}{m^2}, \delta n\,, \Lambda$

3.1 Macro quantities in off-equilibrium

They can be obtained by expectation value of equilibrium !

$l = 0$

$$\int \frac{d^3k\, f_0}{(2\pi)^3 k^0} \lambda_k = \langle \lambda_k \rangle_0 = \frac{\Lambda - 3\Pi}{m^2} \quad \text{or equivalently} \quad \frac{1}{3}\langle \lambda_k k^2 \rangle_0 = \Pi$$

$$\int \frac{d^3k\, f_0}{(2\pi)^3 k^0} \lambda_k E_k = \langle \lambda_k E_k \rangle_0 = \delta n$$

$$\int \frac{d^3k\, f_0}{(2\pi)^3 k^0} \lambda E_k^2 = \left\langle \lambda_k E_k^2 \right\rangle_0 = \Lambda$$

$l = 1$

$$\frac{1}{3!!}\left\langle k^2 \lambda_k^{\langle\mu\rangle} \right\rangle_0 = \delta V^\mu$$

$$\frac{1}{3!!}\left\langle E_k\, k^2 \lambda_k^{\langle\mu\rangle} \right\rangle_0 = W^\mu$$

$l = 2$

$$\frac{1}{5!!}\left\langle k^4 \lambda_k^{\langle\mu\nu\rangle} \right\rangle_0 = \pi^{\mu\nu}$$

Remind: $\lambda^{\langle \mu_1 \mu_2 \cdots \mu_l \rangle}$ *contains information about deviation*

3.2 Equation of State

$$\delta n = \int \frac{d^3k\, \lambda_k f_0}{(2\pi)^3 k^0} E_k = \int \frac{4\pi dk\, \lambda_k f_0}{(2\pi)^3} [\frac{k^3}{3}]' = -\int \frac{4\pi dk}{(2\pi)^3} \frac{k^3}{3} [\lambda_k f_0]'$$

$$= -\int \frac{4\pi dk}{(2\pi)^3} \frac{k^3}{3} \left[\lambda_k' f_0 + \lambda \left(-\beta f_0 \frac{k}{E_k}\right)\right]$$

$$= \beta \int \frac{4\pi k^2 dk\, \lambda_k f_0}{(2\pi)^3 k^0} \frac{k^2}{3} - \int \frac{4\pi k^2 dk f_0}{(2\pi)^3 k^0} \frac{E_k k}{3} \lambda_k' = \quad \beta\Pi - \left\langle \frac{E_k k}{3} \left(\frac{\partial \lambda_k}{\partial k}\right) \right\rangle_0$$

$$\Pi = \delta n\, T + \left\langle \frac{E_k k}{3} \left(\frac{\partial \lambda_k}{\partial k}\right) \right\rangle_0$$ *Off-equilibrium*

$$P_{\mathrm{eq}} = n_{\mathrm{eq}}\, T$$ *Equilibrium*

3.3 Entropy current

$$s^\mu(x) = -\int \frac{d^3k\, k^\mu}{(2\pi)^3 k^0}\,[f_k\ \ln f_k - f_k]$$

$$\approx s^\mu_{\mathrm{eq}}(x) - \int \frac{d^3k\, k^\mu}{(2\pi)^3 k^0}\,\Big[\underbrace{\frac{\delta f}{f_0}\ln f_0}_{\delta s^\mu_1} + \underbrace{\frac{1}{2}\left(\frac{\delta f}{f_0}\right)^2}_{\delta s^\mu_2}\Big]$$

$$\boxed{\begin{aligned}\delta s^\mu_1 &= [-\alpha\delta n + \beta\Lambda]\, u^\mu + [-\alpha\delta V^\mu + \beta W^\mu] \\ \delta s^\mu_2 &= \left[-\frac{1}{2}\left\langle E_k\lambda_k^2\right\rangle_0 - \frac{1}{6}\left\langle k^2 E_k \lambda_k^{\langle\sigma\rangle}\lambda_{k\langle\sigma\rangle}\right\rangle_0 \right. \\ &\qquad \left. - \frac{1}{15}\left\langle k^4 E_k \lambda_k^{\langle\sigma\rho\rangle}\lambda_{k\langle\sigma\rho\rangle}\right\rangle_0\right] u^\mu - \frac{1}{6}\left\langle k^2 E_k \lambda_k \lambda_k^{\langle\mu\rangle}\right\rangle_0\end{aligned}}$$

3.4 Thermo-dynamical stability

Stability condition may be given by

$$\delta s_1^{\mu}\, u_{\mu} \equiv 0 \quad \rightarrow \quad -\alpha\delta n + \beta\Lambda \equiv 0$$

$$\delta n = \beta\Pi - \left\langle \frac{E_k}{3}\left(k\,\frac{\partial\lambda_k}{\partial k}\right)\right\rangle_0$$ *Equation of state*

$$\boxed{\begin{aligned}\Lambda &= -\alpha\Pi + \frac{\alpha}{\beta}\left\langle \frac{E_k}{3}\left(k\,\frac{\partial\lambda_k}{\partial k}\right)\right\rangle_0 \\ &= 3\Pi + \left\langle -\left(1+\frac{\alpha}{3}\right)\lambda k^2 + \frac{\alpha E_k}{3\beta}\left(k\,\frac{\partial\lambda_k}{\partial k}\right)\right\rangle_0\end{aligned}}$$

3.5 δs_2^μ and thermodynamic coefficients

$$\zeta_\Pi \Pi^2 \qquad \zeta_\Pi = \frac{9\langle E_k \lambda_k^2\rangle_0}{\beta\langle k^2\lambda_k\rangle_0^2}$$

$$\zeta_W W^\sigma W_\sigma \qquad \zeta_W = \frac{3\left\langle k^2 E_k\left(-\lambda_k^{\langle\sigma\rangle}\lambda_{k\langle\sigma\rangle}\right)\right\rangle_0}{\beta\left\langle k^2 E_k \lambda_k^{\langle\sigma\rangle}\right\rangle_0\left\langle k^2 E_k\lambda_{k\langle\sigma\rangle}\right\rangle_0}$$

$$-\left[\frac{1}{\beta}\left\langle E_k\lambda_k^2\right\rangle_0 - \frac{1}{3\beta}\left\langle k^2 E_k\left(-\lambda_k^{\langle\sigma\rangle}\lambda_{k\langle\sigma\rangle}\right)\right\rangle_0 + \frac{2}{15\beta}\left\langle k^4 E_k \lambda_k^{\langle\sigma\rho\rangle}\lambda_{k\langle\sigma\rho\rangle}\right\rangle_0\right]\frac{u^\mu}{2T}$$

$$-\frac{1}{6}\left\langle k^2 E_k \lambda_k \lambda_k^{\langle\mu\rangle}\right\rangle_0$$

$$\zeta_\pi \pi^{\sigma\rho}\pi_{\sigma\rho} \qquad \zeta_\pi = \frac{30\left\langle k^4 E_k\lambda_k^{\langle\sigma\rho\rangle}\lambda_{k\langle\sigma\rho\rangle}\right\rangle_0}{\beta\left\langle k^4\lambda_k^{\langle\alpha\beta\rangle}\right\rangle_0\left\langle k^2\lambda_{k\langle\alpha\beta\rangle}\right\rangle_0}$$

This term cannot reduces to ΠW^μ

$$\propto \Pi W^\mu$$

3.6 differential equation of λ_k

Recall $\Lambda = 3\Pi + m^2\langle\lambda_k\rangle_0$

$$\Rightarrow \left\langle \frac{\alpha E_k}{3\beta}\left(k\frac{\partial\lambda_k}{\partial k}\right) - \left(E_k^2 + \frac{\alpha}{3}k^2\right)\lambda_k \right\rangle_0 = 0$$

λ_k *may satisfy the following differential equation*

$$\boxed{\frac{\alpha E_k}{3\beta}\left(k\frac{\partial\lambda_k}{\partial k}\right) - \left(E_k^2 + \frac{\alpha}{3}k^2\right)\lambda_k = 0}$$

4. Summary and concluding remarks

deviation from equilibrium

$$\delta f / f_0 = \sum_{l=0}^{\infty} \lambda_k^{\langle \mu_1 \mu_2 \cdots \mu_l \rangle} k_{<\mu_1} k_{\mu_2} \cdots k_{\mu_l>}$$

Usually imposed to be zero, but it is not obvious

$$\Lambda = \langle \lambda_k E_k^2 \rangle_0, \quad \delta n = \langle \lambda_k E_k \rangle_0, \qquad \Pi = \frac{1}{3} \langle \lambda_k k^2 \rangle_0$$

The lowest possible scheme: $l = 0 \; N_{l=0} = 2$

$$\Pi = \delta n \, T + \left\langle \frac{E_k k}{3} \left(\frac{\partial \lambda_k}{\partial k} \right) \right\rangle_0$$

Equation of State

$$\Lambda = 3\Pi + \left\langle - \left(1 + \frac{\alpha}{3} \right) \lambda k^2 + \frac{\alpha E_k}{3\beta} \left(k \frac{\partial \lambda_k}{\partial k} \right) \right\rangle_0$$

Thermo-dynamical stability

λ_k should satisfy

$$\frac{\alpha E_k}{3\beta} \left(k \frac{\partial \lambda_k}{\partial k} \right) - \left(E_k^2 + \frac{\alpha}{3} k^2 \right) \lambda_k = 0$$

The solution λ_k above can simultaneously satisfy

$$\Lambda = \langle \lambda_k E_k^2 \rangle_0 = 0, \quad \delta n = \langle \lambda_k E_k \rangle_0 = 0 \; ?$$